HOMOLOGY AND STANDARD CONSTRUCTIONS

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Introduction

In ordinary homological algebra, if M is an R-module, the usual way of starting to construct a projective resolution of M is to let F be the free R-module generated by the elements of M and $F \twoheadrightarrow M$ the epimorphism determined by $(m) \mapsto m$. One then takes the kernel of $F \twoheadrightarrow M$ and continues the process. But notice that in the construction of $F \twoheadrightarrow M$ a lot of structure is customarily overlooked. F is actually a functor MG of M, $F \twoheadrightarrow M$ is an instance of a natural transformation $G \twoheadrightarrow$ (identity functor); there is also a "comultiplication" $G \twoheadrightarrow GG$ which is a little less evident. The functor G, equipped with these structures, is an example of what is called a standard construction or "cotriple".

In this paper we start with a category $\mathbf{C}$, a cotriple $\mathbf{G}$ in $\mathbf{C}$, and show how resolutions and derived functors or homology can be constructed by means of this tool alone. The category $\mathbf{C}$ will be non-abelian in general (note that even for modules the cotriple employed fails to respect the additive structure of the category), and the coefficients will consist of an arbitrary functor $E \colon \mathbf{C} \twoheadrightarrow \mathscr{A}$, where $\mathscr{A}$ is an abelian category. For ordinary homology and cohomology theories, E will be tensoring, homming or deriving with or into a module of some kind.

To summarize the contents of the paper: In Section 1 we define the derived functors and give several examples of categories with cotriples. In Section 2 we study the derived functors $H_n(\ ,E)_{\mathbf{G}}$ as functors on $\mathbf{C}$ and give several of their properties. In Section 3 we fix a first variable $X \in \mathbf{C}$ and study $H_n(X,\ )_{\mathbf{G}}$ as a functor of the abelian variable E. As such it admits a simple axiomatic characterization. Section 4 considers the case in which $\mathbf{C}$ is additive and shows that the general theory can always, in effect, be reduced to that case. In Section 5 we study the relation between cotriples and projective classes (defined—essentially—by Eilenberg-Moore [Eilenberg & Moore (1965)]) and show that the homology only depends on the projective class defined by the cotriple. Sections 6–9 are concerned largely with various special properties that these derived functors possess in well known algebraic categories (groups, modules, algebras, . . .). In Section 10 we consider the problem of defining a cotriple to produce a given projective class (in a sense, the converse problem to that studied in Section 5) by means of "models". We also compare the results with other theories of derived functors based on models. Section 11 is concerned with some technical items on acyclic models.

Before beginning the actual homology theory, we give some basic definitions concerning the simplicial objects which will be used. Let $\mathbf{G} = (G, \varepsilon, \delta)$ be a cotriple in $\mathbf{C}$, that is,

$$\mathbf{C} \xrightarrow{G} \mathbf{C}$$

$$G \xrightarrow{\varepsilon} \mathbf{C} \quad \text{and} \quad G \xrightarrow{\delta} GG$$

and the unitary and associative laws hold, as given in the Introduction to this volume. (Note that here and throughout we identify identity maps with the corresponding objects; thus $\mathbf{C}$ denotes the identity functor $\mathbf{C} \to \mathbf{C}$.) If X is an object in $\mathbf{C}$, the following is an augmented simplicial object in $\mathbf{C}$:

$$X \xleftarrow{\varepsilon_0} XG \underset{\varepsilon_1}{\overset{\varepsilon_0}{\underset{\delta_0}{\rightleftarrows}}} XG^2 \;\substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow} \cdots \;\substack{\longleftarrow\\[-2pt]\vdots\\[-2pt]\longrightarrow} XG^{n+1} \;\substack{\longleftarrow\\[-2pt]\vdots\\[-2pt]\longrightarrow} \cdots$$

XG^{n+1} is the n-dimensional component,

$$\varepsilon_i = G^i \varepsilon\, G^{n-i} \colon G^{n+1} \to G^n \quad \text{and} \quad \delta_i = G^i \delta G^{n-i} \colon G^{n+1} \to G^{n+2}$$

for $0 \le i \le n$, and the usual simplicial identities hold:

$$\varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i \quad \text{for} \quad i \le j$$
$$\delta_i \delta_j = \delta_{j-1} \delta_i \quad \text{for} \quad i < j$$

$$\delta_i \varepsilon_j = \begin{cases} \varepsilon_{j-1} \delta_i & \text{for} \quad i < j-1 \\ \text{identity} & \text{for} \quad i = j-1 \ \text{and} \ i = j \\ \varepsilon_j \delta_{i-1} & \text{for} \quad i > j. \end{cases}$$

(composition is from left to right).

If X admits a map $s \colon X \to XG$ such that $s \circ X\varepsilon = X$ (such X are called $\mathbf{G}$-projective, see (2.1)), then the above simplicial object develops a **contraction**

$$X \xrightarrow{h_{-1}} XG \xrightarrow{h_0} XG^2 \longrightarrow \cdots \longrightarrow XG^{n+1} \xrightarrow{h_n} \cdots$$

namely $h_n = sG^{n+1}$. These operators satisfy the equations

$$h_n \varepsilon_0 = XG^{n+1} \quad \text{and} \quad h_n \varepsilon_i = \varepsilon_{i-1} h_{n-1}$$

for $0 < i \le n+1$ and $n \ge -1$. They express the fact that the simplicial object $(XG^{n+1})_{n \ge 0}$ is homotopically equivalent to the constant simplicial object which has X in all dimensions.

If $(X_n)_{n \ge -1}$ is a simplicial set with such a contraction, we conclude $\Pi_n(X) = 0$ for $n > 0$, and $\Pi_0(X) = X_{-1}$.

On the other hand, if $E \colon \mathbf{C} \to \mathscr{A}$ is a functor into any other category and E possesses a natural transformation $\vartheta \colon E \to GE$ such that $\vartheta \circ \varepsilon E = E$, then $(XG^{n+1}E)_{n \ge -1}$ also has a contraction

$$XE \xrightarrow{h_{-1}} XGE \xrightarrow{h_0} XG^2E \longrightarrow \cdots \longrightarrow XG^{n+1}E \xrightarrow{h_n} \cdots$$

Here $h_n = XG^{n+1}\vartheta$ and the identities satisfied are a little different (This is a "right" homotopy [Kleisli (1967)]):

$$h_n\varepsilon_i = \varepsilon_i h_{n-1} \quad \text{and} \quad h_n\varepsilon_{n+1} = XG^{n+1}E$$

for $0 \le i \le n$ and $n \ge -1$. Both here and above some equations involving degeneracies also hold, but our concern is usually with homology so we omit them.

If the functor E takes values in an abelian category, then as follows from a well known theorem of J.C. Moore [Moore (1956)] the homotopy in any sense of $(XG^{n+1}E)_{n\ge 0}$ is the same as the homology of the associated chain complex

$$0 \longleftarrow XGE \xleftarrow{\partial_1} XG^2E \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_n} XG^{n+1}E \longleftarrow \cdots$$

where $\partial_n = \sum(-1)^i\varepsilon_i E$. If there is a contraction, $H_n = 0$ for $n > 0$, $H_0 = XE$.

1. Definition of the homology theory $H_n(X, E)_{\mathbf{G}}$

Let $X \in \mathbf{C}$, let $\mathbf{G} = (G, \varepsilon, \delta)$ be a cotriple in $\mathbf{C}$, and let $E \colon \mathbf{C} \to \mathscr{A}$ be a functor into an abelian category. Applying E to $(XG^{n+1})_{n\ge -1}$ we get an augmented simplicial object in $\mathscr{A}$:

$$XE \longleftarrow XGE \rightrightarrows XG^2E \mathrel{\substack{\longrightarrow\\[-0.6em]\longrightarrow\\[-0.6em]\longrightarrow}} \cdots \mathrel{\substack{\longrightarrow\\[-0.6em]\vdots\\[-0.6em]\longrightarrow}} XG^{n+1}E \mathrel{\substack{\longrightarrow\\[-0.6em]\vdots\\[-0.6em]\longrightarrow}} \cdots$$

The homotopy of this simplicial object, or what is the same thing by Moore's theorem, the homology of the associated chain complex

$$0 \longleftarrow XGE \xleftarrow{\partial_1} XG^2E \xleftarrow{\partial_2} \cdots$$

is denoted by $H_n(X, E)_{\mathbf{G}}$, for $n \ge 0$. These are the **homology groups** (objects) **of X with coefficients in E relative to the cotriple $\mathbf{G}$**. Often $\mathbf{G}$ is omitted from the notation if it is clear from the context.

The homology is functorial with respect to maps $X \to X_1$ in $\mathbf{C}$ and natural transformations of the coefficient functors $E \to E_1$.

A natural transformation (**augmentation**)

$$H_0(\ , E)_{\mathbf{G}} \xrightarrow{\lambda = \lambda E} E$$

is defined by the fact that H_0 is a cokernel:

$$XG^2E \xrightarrow{\partial_1} XGE \longrightarrow H_0(X, E) \longrightarrow 0$$

with ∂_0 going down to XE and λ from $H_0(X,E)$ to XE.

$\lambda(H_0(\ ,E))$ and $H_0(\ ,\lambda E)$ coincide since they both fit in the diagram

$$
\begin{array}{ccccccc}
H_0(XG^2, E) & \longrightarrow & H_0(XG, E) & \longrightarrow & H_0(X, H_0(\ ,E)) & \longrightarrow & 0 \\
\downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda} & & \downarrow & & \\
XG^2E & \longrightarrow & XGE & \longrightarrow & H_0(X, E) & \longrightarrow & 0
\end{array}
$$

Thus λ can be viewed as a reflection into the subcategory of all functors $E\colon \mathbf{C} \twoheadrightarrow \mathscr{A}$ with $\lambda\colon H_0(\ ,E) \xrightarrow{\cong} E$. These are the functors which transform $XG^2 \rightrightarrows XG \longrightarrow X$ into a coequalizer diagram in $\mathscr{A}$, for all $X \in \mathbf{C}$, a sort of right exactness property.

The following variations occur. If, dually, $\mathbf{T} = (T, \eta, \mu)$ is a triple in $\mathbf{C}$ and $E\colon \mathbf{C} \twoheadrightarrow \mathscr{A}$ is a coefficient functor, **cohomology groups** $H^n(X, E)_{\mathbf{T}}$, for $n \geq 0$, are defined by means of the cochain complex

$$
0 \longrightarrow XTE \xrightarrow{d^1} XT^2E \xrightarrow{d^2} \cdots \longrightarrow XT^{n+1}E \xrightarrow{d^n} \cdots
$$

where $d^n = \sum(-1)^i X\eta_i E$ for $0 \leq i \leq n$, and $\eta_i = T^i\eta T^{n-i}$.

If $\mathbf{G} = (G, \varepsilon, \delta)$ is a cotriple and $E\colon \mathbf{C}^{\mathrm{op}} \twoheadrightarrow \mathscr{A}$ is a functor (or $E\colon \mathbf{C} \twoheadrightarrow \mathscr{A}$ is contravariant), the complex would take the form

$$
0 \longrightarrow XGE \longrightarrow XG^2E \longrightarrow \cdots \longrightarrow XG^{n+1}E \longrightarrow \cdots
$$

In effect this is cohomology with respect to the triple $\mathbf{G}^{\mathrm{op}}$ in the dual category. However, we write the theory as $H^n(X, E)_{\mathbf{G}}$.

For the most part we will only state theorems about the cotriple-covariant functor situation and leave duals to the reader. Usually cotriples arise from adjoint functors, although another method of construction will be essayed in Section 10. If $F\colon \mathbf{A} \twoheadrightarrow \mathbf{C}$ is left adjoint to $U\colon \mathbf{C} \twoheadrightarrow \mathbf{A}$, there are well know natural transformations $\eta\colon \mathbf{A} \twoheadrightarrow FU$ and $\varepsilon\colon UF \twoheadrightarrow \mathbf{C}$. If we set $G = UF$, we have $\varepsilon\colon G \twoheadrightarrow \mathbf{C}$, and if we set $\delta = U\eta F$, then $\delta\colon G \twoheadrightarrow G^2$.[a] The relations obeyed by η and ε

$$
\begin{array}{ccc}
F \xrightarrow{\ \eta F\ } FUF & \qquad\qquad & U \xrightarrow{\ U\eta\ } UFU \\
{\scriptstyle =}\searrow \quad \downarrow{\scriptstyle F\varepsilon} & & {\scriptstyle =}\searrow \quad \downarrow{\scriptstyle \varepsilon U} \\
\qquad F & & \qquad U
\end{array}
$$

imply that $\mathbf{G} = (G, \varepsilon, \delta)$ is a cotriple in $\mathbf{C}$. This fact was first recognized by Huber [Huber (1961)].

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[a]Editor's footnote: The $\mathbf{C}$ here and in the sentence before were $\mathbf{B}$ in the original, but as this was a mistake they are corrected now.

1.1 ADDITIVE EXAMPLE: HOMOLOGY OF MODULES. Let R-**Mod** be the category of (left) R-modules. Let $\mathbf{G} = (G, \varepsilon, \delta)$ be the cotriple generated by the adjoint pair

$$R\text{-}\mathbf{Mod} \xrightarrow{\quad G \quad} R\text{-}\mathbf{Mod}$$

with U down-left and F down-right to **Set**.

U is the usual underlying set functor, $F \dashv U$ is the free R-module functor. Thus we have $MG = R(M)$, the free R-module with the elements of M as basis, and the counit $M\varepsilon \colon Mg \twoheadrightarrow M$ is the map which takes each basis element into the same element in M (just the usual way of starting to construct an R-free resolution of M). The comultiplication $M\delta \colon MG \twoheadrightarrow MG^2$ we leave to the reader.

Later we shall show that the complex

$$0 \longleftarrow M \xleftarrow{\ \partial_0\ } MG \xleftarrow{\ \partial_1\ } MG^2 \longleftarrow \cdots \xleftarrow{\ \partial_n\ } MG^{n+1} \longleftarrow \cdots$$

where $\partial_n = \sum (-1)^i M\varepsilon_i$ for $0 \le i \le n$, is an R-free resolution of M (the only issue is exactness). Taking as coefficient functors $ME = A \otimes_R M$ or $ME = \mathrm{Hom}_R(M, A)$, we obtain $H_n(M, A \otimes_R\ )$ and $H^n(M, \mathrm{Hom}_R(\ , A))$ as n-th homology or cohomology of

$$0 \longleftarrow A \otimes_R MG \longleftarrow A \otimes_R MG^2 \longleftarrow \cdots$$

$$0 \longrightarrow \mathrm{Hom}_R(MG, A) \longrightarrow \mathrm{Hom}_R(MG^2, A) \longrightarrow \cdots$$

That is, $H_n = \mathrm{Tor}_n^R(A, M)$ and $H^n = \mathrm{Ext}_R^n(M, A)$.

Since R-modules are an additive category and the coefficient functors considered were additive, we could form the alternating sum of the face operators to obtain a chain complex in R-**Mod** before applying the coefficient functor.

As another example of this we mention the Eckmann-Hilton homotopy groups $\Pi_n(M, N)$ (as re-indexed in accordance with [Huber (1961)]). These are the homology groups of the complex

$$0 \longleftarrow \mathrm{Hom}_R(M, NG) \longleftarrow \mathrm{Hom}_R(M, NG^2) \longleftarrow \cdots$$

Of course, in these examples the homology should have a subscript $\mathbf{G}$ to indicate that the cotriple relative to the underlying category of sets was used to construct the resolution. Other underlying categories and cotriples are possible. For example, if

$$K \xrightarrow{\ \varphi\ } R$$

is a ring map, we get an adjoint pair

$$R\text{-}\mathbf{Mod} \xrightarrow{\quad G_\varphi \quad} R\text{-}\mathbf{Mod}$$

with down-left arrow to K-**Mod** and $\otimes_K R$ down-right.

where the underlying [functor] is restriction of operators to K by means of φ. We have $MG_\varphi = M \otimes_K R$. The standard resolution is

$$M \longleftarrow M \otimes_K R \Longleftarrow M \otimes_K R \otimes_K R \Lleftarrow \cdots$$

Using the above coefficient functors we will find that the homology and cohomology are Hochschild's K-relative Tor and Ext [Hochschild (1956)]:

$$H_n(M, A \otimes_R \ ) = \mathrm{Tor}_n^\varphi(A, M)$$
$$H^n(M, \mathrm{Hom}_R(\ , A)) = \mathrm{Ext}_\varphi^n(M, A)$$

Hochschild actually considered a subring $K \rightarrowtail R$ and wrote $\mathrm{Tor}^{(R,K)}$, etc.

We now turn to homology of groups and algebras. A useful device in the non-additive generalizations of homology theory is the *comma category* $(\mathbf{C}, X)$ of all **objects** (of a given category $\mathbf{C}$) **over a fixed object** X. That is, an object of $(\mathbf{C}, X)$ is a map $C \rightarrow X$, and a map of $(\mathbf{C}, X)$ is a commutative triangle
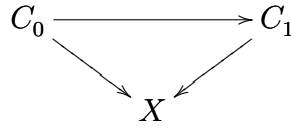
A cotriple $\mathbf{G} = (G, \varepsilon, \delta)$ in $\mathbf{C}$ naturally operates in $(\mathbf{C}, X)$ as well. The resulting cotriple $(\mathbf{G}, X)$ has

$$(C \xrightarrow{p} X)(G, X) = CG \xrightarrow{C\varepsilon} C \xrightarrow{p} X$$

The standard $(\mathbf{G}, X)$-resolution of an object $C \rightarrow X$ over X comes out in the form
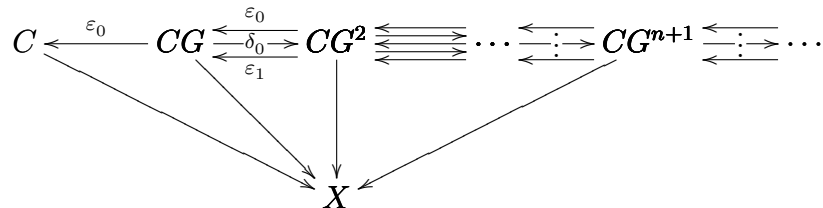
In other words, the usual faces and degeneracies turn out to be maps over X.

Homology groups $H_n(C, E)_{(\mathbf{G}, X)}$ are then defined, when $E \colon (\mathbf{C}, X) \rightarrow \mathscr{A}$ is a coefficient functor. We could write, with greater precision, $H_n(p, E)_{(\mathbf{G}, X)}$, or with less, $H_n(C, E)_X$ or $H_n(C, E)$, leaving X understood.

Usually the coefficient functors involve a module over the terminal object X. This can be treated as a module over all the objects of $(\mathbf{C}, X)$ simultaneously, by pullback via the structural maps to X. For example, derivations or differentials with values in an X-module become functors on the category of all algebras over X. This is the way in which homology and cohomology of algebras arise.

1.2 HOMOLOGY OF GROUPS. Let $\mathbf{Gr}$ be the category of groups and $\mathbf{G}$ the cotriple arising from

$$\mathbf{Gr} \xrightarrow{\quad G \quad} \mathbf{Gr}$$
$$U \searrow \qquad \nearrow F$$
$$\mathbf{Set}$$

Thus ΠG is the free group on the underlying set of Π, and the counit $\Pi G \twoheadrightarrow \Pi$ is the natural surjection of the free group onto Π.

If $W \twoheadrightarrow \Pi$ is a group over Π and M is a left Π-module, a **derivation** $f: W \twoheadrightarrow M$ **(over Π)** is

a function such that $(ww')f = w \cdot w'f + wf$ ($W \twoheadrightarrow \Pi$ allows W to act on M). The abelian group of such derivations, $\mathrm{Der}(W, M)_\Pi$, gives a functor $(\mathbf{Gr}, \Pi)^{\mathrm{op}} \to \mathbf{Ab}$. We define the cohomology of $W \twoheadrightarrow \Pi$ with coefficients in M, $H^n(W, M)_\Pi$ (relative to $\mathbf{G}$) as the cohomology of the cochain complex

$$0 \longrightarrow \mathrm{Der}(WG, M)_\Pi \longrightarrow \mathrm{Der}(WG^2, M)_\Pi \longrightarrow \cdots \longrightarrow \mathrm{Der}(WG^{n+1}, M)_\Pi \longrightarrow \cdots$$

It is known that this theory coincides with Eilenberg-Mac Lane cohomology except for a shift in dimension [Barr & Beck (1966)]

$$H^n(W, M)_\Pi \xrightarrow{\cong} \begin{cases} \mathrm{Der}(W, M)_\Pi & \text{for } n = 0 \\ H^{n+1}_{E-M}(W, M) & \text{for } n > 0 \end{cases}$$

Derivations $W \twoheadrightarrow M$ are represented by a Π-module of **differentials of W (over Π)** which we write as $\mathrm{Diff}_\Pi(W)$:

$$\mathrm{Der}(W, M)\Pi \cong \mathrm{Hom}_W(IW, M)$$
$$= \mathrm{Hom}_\Pi(\mathbf{Z}\Pi \otimes_W IW, M)$$

Hence $\mathrm{Diff}_\Pi(W) = \mathbf{Z}\Pi \otimes_W IW$. (It is well known that the augmentation ideal $IW = \ker(\mathbf{Z}W \twoheadrightarrow \mathbf{Z})$ represents derivations of W into W-modules [Cartan & Eilenberg (1956), Mac Lane (1963)]. This is fudged by $\mathbf{Z}\Pi \otimes_W$ to represent derivations into Π-modules.)

The **homology of $W \twoheadrightarrow \Pi$ with coefficients in a right Π-module** M is defined as the homology of

$$0 \longleftarrow M \otimes_{\mathbf{Z}\Pi} \mathrm{Diff}_\Pi(WG) \longleftarrow M \otimes_{\mathbf{Z}\Pi} \mathrm{Diff}_\Pi(WG^2) \longleftarrow \cdots \longleftarrow M \otimes_{\mathbf{Z}\Pi} \mathrm{Diff}_\Pi(WG^{n+1}) \longleftarrow \cdots$$

Then

$$H_n(W, M)_\Pi \xrightarrow{\cong} \begin{cases} M \otimes_{\mathbf{Z}\Pi} \mathrm{Diff}_\Pi(W) = M \otimes_{\mathbf{Z}\Pi} IW & \text{for } n = 0 \\ H^{E-M}_{n+1}(W, M) & \text{for } n > 0 \end{cases}$$

This is because $(\mathrm{Diff}_\Pi(WG^{n+1}))_{n \geq -1}$ is a Π-free resolution of $\mathrm{Diff}_\Pi(W)$, and as [Cartan & Eilenberg (1956), Mac Lane (1963)] show, the Eilenberg-Mac Lane homology can be identified with $\mathrm{Tor}_{n+1}^{\mathbf{Z}W}(\mathbf{Z}, N) = \mathrm{Tor}_n^{\mathbf{Z}W}(IW, N)$. Π-Freeness is because $\mathrm{Diff}_\Pi(WG) = \mathbf{Z}\Pi \otimes_W I(WG)$, and $I(WG)$ is well known to be WG-free. As for acyclicity, the cohomology of

$$0 \longrightarrow \mathrm{Hom}_\Pi(\mathrm{Diff}_\Pi(W), Q) \longrightarrow \mathrm{Hom}_\Pi(\mathrm{Diff}_\Pi(WG), Q) \longrightarrow \cdots$$

is zero in all dimensions ≥ -1, if Q is an injective Π-module; this is true because the cohomology agrees with the Eilenberg-Mac Lane theory, which vanishes on injective coefficient modules. A direct acyclic-models proof of the coincidence of the homology theories can also be given.

As special cases note: if Π is regarded as a group over Π by means of the identity map $\Pi \to \Pi$, the $H^n(\Pi, M)_\Pi$ and $H_n(\Pi, M)_\Pi$ are the ordinary (co-)homology groups of Π with coefficients in a Π-module. On the other hand, if $\Pi = 1$, any W can be considered as a group over Π. Since a 1-module is just an abelian group, $\mathrm{Diff}_1(W) = W/[W, W]$, [which is] W abelianized, i.e. with its commutator subgroup divided out. The (co-)homology is that of W with coefficients in a trivial module.

REMARK. [Beck (1967), Barr & Beck (1966)] Via interpretation as split extensions, Π-modules can be identified with the abelian group objects in the category $(\mathbf{Gr}, \Pi)$. $\mathrm{Der}(W, M)_\Pi$ is then the abelian group of maps in $(\mathbf{Gr}, \Pi)$:

Diff_Π is just the free abelian group functor, that is, the left adjoint of the forgetful functor

$$(\mathbf{Gr}, \Pi) \longleftarrow \mathbf{Ab}(\mathbf{Gr}, \Pi) = \Pi\text{-}\mathbf{Mod}$$

where $\mathbf{Ab}(\mathbf{Gr}, \Pi)$ denotes the abelian groups in $(\mathbf{Gr}, \Pi)$.

For general triple cohomology this interpretation is essential. In particular, the analogue of Diff exists for any category tripleable over $\mathbf{Set}$, provided the triple has a rank in the sense of [Linton (1966a)].

For the next example we need the comma category $(X, \mathbf{C})$ of objects and maps in $\mathbf{C}$ *under* X. An object of this category is a map $X \to Y$, a map is a commutative triangle

Assuming $\mathbf{C}$ has coproducts $X * Y$, a cotriple $\mathbf{G} = (G, \varepsilon, \delta)$ in $\mathbf{C}$ naturally induces a

cotriple $(X, \mathbf{G}) = ((X, G), \ldots)$ in $(X, \mathbf{C})$:

$$(X \xrightarrow{f} Y)(X, G) = X \longrightarrow X * YG \qquad \text{(coproduct injection)}$$

$(X \longrightarrow Y)(X, \varepsilon) =$

$(X \longrightarrow Y)(X, \delta) =$

where $j : YG \rightarrow X * YG$ is a coproduct injection.

Actually, the coproduct $X * (\ )$ defines an adjoint pair of functors $(X, \mathbf{C}) \rightarrow \mathbf{C} \rightarrow (X, \mathbf{C})$; the right adjoint is $(X \rightarrow C) \mapsto C$, the left adjoint is $C \mapsto (X \rightarrow X * C)$. By a general argument [Huber (1961)], the composition

$$(X, \mathbf{C}) \longrightarrow \mathbf{C} \xrightarrow{G} \mathbf{C} \xrightarrow{X * (\ )} (X, \mathbf{C})$$

is then a cotriple in $(X, \mathbf{C})$, namely $(X, \mathbf{G})$.

Replacing $(X, \mathbf{C}) \rightarrow \mathbf{C} \rightarrow (X, \mathbf{C})$ by an arbitrary adjoint pair and specializing $\mathbf{G}$ to the identity cotriple proves the remark preceding (1.1).

Homology and cohomology relative to the cotriple $(X, \mathbf{G})$ will be studied in more detail in Section 8. This cotriple enters in a rather mild way into:

1.3 Homology of commutative rings and algebras. Let **Comm** be the category of commutative rings. For $A \in \mathbf{Comm}$ let $(A, \mathbf{Comm})$ be the category of commutative rings under A, that is, maps $A \rightarrow B \in \mathbf{Comm}$. Thus $(A, \mathbf{Comm})$ is our notation for the category of commutative A-algebras. We review the notions of differentials and derivations in this category.

For the same reason as in the category of groups we place ourselves in a category of algebras *over* a fixed commutative ring D, that is, in a double comma category $(A, \mathbf{Comm}, D)$; here an object is an A-algebra $A \rightarrow B$ equipped with a map $B \rightarrow D$, and a map is a commutative diagram

If M is a D-module, an **A-derivation** $B \rightarrow M$ is an A-linear function satisfying $(bb')f = b' \cdot bf + b \cdot b'f$, where $B \in (A, \mathbf{Comm}, D)$ and A and B act on M via the given

maps $A \twoheadrightarrow B \twoheadrightarrow D$. Such modules of derivations define a functor

$$(A, \mathbf{Comm}, D)^{\mathrm{op}} \xrightarrow{\ A\text{-}\mathrm{Der}(\ ,M)_D\ } D\text{-}\mathbf{Mod}$$

This eventually gives rise to cohomology.

As is well known, any A-derivation $B \twoheadrightarrow M$, where M is a B-module, factors uniquely through a B-module map

$$
\begin{array}{ccc}
 & B & \\
{\scriptstyle d}\swarrow & & \searrow \\
\Omega^1_{B/A} & \text{-\,-\,-\,-\,-\,-} \rightarrow & M
\end{array}
$$

where $\Omega^1_{B/A}$ is the B**-module of A-differentials of** B, and d is the *universal* such derivation. $\Omega^1_{B/A}$ can be viewed as I/I^2 where $I = \ker(B \otimes_A B \twoheadrightarrow B)$ and $db = b \otimes 1 - 1 \otimes b$, or as the free B-module on symbols db modulo $d(b+b') = db + db'$ as well as $d(ab) = a \cdot db$ and $d(bb') = b' \cdot db + b \cdot db'$ [Lichtenbaum & Schlessinger (1967), Grothendieck & Dieudonné (1964)]. [A] universal [object] for A-derivations of $B \twoheadrightarrow M$, where M is a D-module, is then

$$\mathrm{Diff}_D(A \twoheadrightarrow B) = \Omega^1_{B/A} \otimes_B D$$

The functor which is usually used as coefficients for homology is

$$(A, \mathbf{Comm}, D) \xrightarrow{\ \mathrm{Diff}_D(A\ \twoheadrightarrow\ (\ ))\otimes_D M = \Omega^1_{(\ )/A}\otimes_{(\ )} M\ } D\text{-}\mathbf{Mod}$$

There are two natural ways of defining homology in the category of A-algebras (over D), depending on the choice of cotriple, or equivalently, choice of the underlying category.

First let $\mathbf{G} = (G, \varepsilon, \delta)$ be the cotriple in the category of commutative rings arising from the adjoint pair

$$
\begin{array}{ccc}
\mathbf{Comm} & \xrightarrow{\quad G \quad} & \mathbf{Comm} \\
{\scriptstyle U}\searrow & & \nearrow{\scriptstyle F} \\
 & \mathbf{Set} &
\end{array}
$$

Then $CG = \mathbf{Z}[C]$, the polynomial ring with the elements of C as variables; the counit $CG \twoheadrightarrow C$ is the map defined by sending the variable c [to the element] $c \in C$. This cotriple operates in $(\mathbf{Comm}, D)$ in the natural fashion described before (1.2).

Now consider the category $(A, \mathbf{Comm})$ of commutative A-algebras. According to the remarks preceding this section, $\mathbf{G}$ gives rise to a cotriple $(A, \mathbf{G})$ in this category. Since the coproduct in the category $\mathbf{Comm}$ is $A \otimes_{\mathbf{Z}} B$, we have

$$
\begin{aligned}
(A \twoheadrightarrow C)(A, G) = A &\twoheadrightarrow A \otimes_{\mathbf{Z}} CG \\
&= A \twoheadrightarrow A \otimes_{\mathbf{Z}} \mathbf{Z}[C] \\
&= A \twoheadrightarrow A[C]
\end{aligned}
$$

the polynomial A-algebra with the elements of C as variables. This cotriple is just that which is induced by the underlying set and free A-algebra functors

$$(A, \textbf{Comm}) \xrightarrow{\;(A,G)\;} (A, \textbf{Comm})$$

$$U \searrow \qquad \nearrow F$$

$$\textbf{Set}$$

Furthermore, $(A, \textbf{G})$ operates in the category of A-algebras over D, $(A, \textbf{Comm}, D)$, the values of $(A, \textbf{G}, D)$ being given by:

$$\begin{pmatrix} A \\ \downarrow \\ C \\ \downarrow \\ D \end{pmatrix} \longmapsto \begin{pmatrix} A \\ \downarrow \\ A[C] \\ \downarrow \\ D \end{pmatrix}$$

The counit is:

$$\begin{array}{ccc} & A & \\ & \swarrow \quad \searrow & \\ A[C] & \xrightarrow{\;\;\varepsilon\;\;} & C \\ & \searrow \quad \swarrow & \\ & D & \end{array}$$

If M is a D-module we thus have homology and cohomology D-modules $H_n(C, M)$ and $H^n(C, M)$ for $n \geq 0$, writing simply C for an A-algebra over D. These are defined by

$$H_n(C, M) = H_n[(\mathrm{Diff}_D(C(A, G)^{p+1}) \otimes_D M)_{p \geq 0}]$$
$$= H_n[(\Omega^1_{A[\cdots[C]\cdots]/A} \otimes_A M)_{p \geq 0}]$$

where there are $p + 1$ applications of the A-polynomial operation to C in dimension p, and by

$$H^n(C, M) = H^n[(A\text{-}\mathrm{Der}_D(C(A, G)^{p+1}, M))_{p \geq 0}]$$
$$= H^n[(A\text{-}\mathrm{Der}_D(A[\cdots[C]\cdots], M))_{p \geq 0}]$$

again with $p + 1$ $A[\ ]$'s.

This homology theory of commutative algebras over D coincides with those considered in [André (1967), Quillen (1967)]; of course, one generally simplifies the setting slightly

by taking $C = D$ above. Both of these papers contain proofs that the cotriple theory coincides with theirs. The homology theory of [Lichtenbaum & Schlessinger (1967)] also agrees.

This theory, however it is described, is called the **"absolute"** homology theory of commutative algebras. The term arises as a reference to the underlying category which is involved, namely that of sets; no underlying object functor could forget more structure. But it also seems germane to consider so-called **relative** homology theories of algebras for which the underlying category is something else, usually a category of modules.

As an example of this, consider the homology theory in $(A, \mathbf{Comm})$ comming from the adjoint functors

$$(A, \mathbf{Comm}) \xrightarrow{\quad G_A \quad} (A, \mathbf{Comm})$$

$$U \searrow \qquad \nearrow F$$

$$A - \mathbf{Mod}$$

That is

$$(A \twoheadrightarrow C)G_A = A + C + \frac{C \otimes_A C}{S_2} + \frac{C \otimes_A C \otimes_A C}{S_3} + \cdots$$

the symmetric A-algebra on C (the S's are the symmetric groups). Note that this cotriple is not of the form $(A, \mathbf{G})$ for any cotriple $\mathbf{G}$ on the category of commutative rings. Exactly as above we now have homology and cohomology groups

$$H_n(C, M) = H_n[(\mathrm{Diff}_D(CG_A^{p+1}) \otimes_D M)_{p \geq 0}]$$

$$= H_n[(\Omega^1_{CG_A^{p+1}/A} \otimes_D M)_{p \geq 0}]$$

$$H^n(C, M) = H^n[(A\text{-}\mathrm{Der}(CG_A^{p+1}, M)_D)_{p \geq 0}]$$

where M is a D-module, and we are writing C instead of $A \twoheadrightarrow C$ for an A-algebra.

These two cohomology theories should really be distinguished by indicating the cotriple used to define them:

$$H^n(C, M)_{(A, \mathbf{G})} = \text{ absolute theory, relative to sets}$$

$$H^n(C, M)_{\mathbf{G}_A} = \text{ theory relative to } A\text{-modules}$$

The following is an indication of the difference between them: if $C = D$ and M is a C-module, then $H^1(C, M)_{(A, \mathbf{G})}$ classifies commutative A-algebra extensions $E \twoheadrightarrow C$ such that $I = \ker(E \twoheadrightarrow C)$ is an ideal of E with $I^2 = 0$, and such that there exists a lifting of the counit

$$
\begin{array}{ccc}
 & & E \\
 & \nearrow & \downarrow \\
C(A, G) & \longrightarrow & C
\end{array}
$$

$H^1(C, M)_{\mathsf{G}_A}$ classifies those extensions with kernel of square zero that have liftings
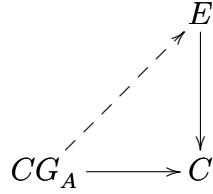
The absolute lifting condition is equivalent to the existence of a set section of $E \twoheadrightarrow C$, i.e. to surjectivity, the A-relative condition to the existence of an A-linear splitting of $E \twoheadrightarrow C$, as one can easily check. The relative theory is thus insensitive to purely A-linear phenomena, while the absolute theory takes all the structure into account. (We refer to [Beck (1967)] for details on classification of extensions).

The A-relative cohomology theory has been studied but little. Harrison has given an A-relative theory in [Harrison (1962)] (A was a ground field but his formulas are meaningful for any commutative ring). Barr [Barr (1968)] has proved that

$$H^n(C, M) \cong \begin{cases} \operatorname{Der}(C, M) & \text{for } n = 0 \\ \operatorname{Harr}^{n+1}(C, M) & \text{for } n > 0 \end{cases}$$

if A is a field of characteristic zero.

1.4 HOMOLOGY OF ASSOCIATIVE K-ALGEBRAS. Let G_K be the cotriple relative to the underlying category of K-modules:
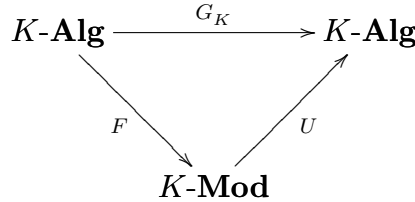
Thus if Λ is an associative algebra with unit over the commutative ring K, then

$$\Lambda G_K = K + \Lambda + \Lambda \otimes \Lambda + \cdots$$

the K-tensor algebra.

If $\Gamma \twoheadrightarrow \Lambda$ and M is a Λ-Λ-bimodule, we define $H^n(\Gamma, M)_\Lambda$ as the cohomology of the cosimplicial object

$$0 \longrightarrow \operatorname{Der}(\Gamma G_K, M)_\Lambda \rightrightarrows \operatorname{Der}(\Gamma G_K^2, M)_\Lambda \Rrightarrow \cdots \longrightarrow \operatorname{Der}(\Gamma G_K^{n+1}, M)_\Lambda \longrightarrow \cdots$$

It is known that this coincides with Hochschild cohomology [Barr (1966), Barr & Beck (1966)]:

$$H^n(\Gamma, M)_\Lambda \xrightarrow{\;\cong\;} \begin{cases} \operatorname{Der}(\Gamma, M) & \text{for } n = 0 \\ \operatorname{Hoch}^{n+1}(\Gamma, M) & \text{for } n > 0 \end{cases}$$

The universal object for K-linear derivations $\Gamma \twoheadrightarrow M$, where M is a two-sided Λ-module, is

$$\mathrm{Diff}_\Lambda(\Gamma) = \mathrm{Diff}_\Gamma(\Gamma) \otimes_{\Gamma^e} \Lambda^e = \Lambda \otimes_\Gamma J\Gamma \otimes_\Gamma \Lambda$$

where $J\Gamma$ is the kernel of the multiplication $\Gamma^e = \Gamma \otimes_K \Gamma^{\mathrm{op}} \twoheadrightarrow \Gamma$ and represents derivations of Γ into Γ-modules [Cartan & Eilenberg (1956), Mac Lane (1963)]. The **homology** of $\Gamma \twoheadrightarrow \Lambda$ with coefficients in M is defined as the homology of the complex

$$0 \longleftarrow \mathrm{Diff}_\Lambda(\Gamma G_K) \otimes_{\Lambda^e} M \xleftarrow{\ \partial_1\ } \mathrm{Diff}_\Lambda(\Gamma G_K^2) \otimes_{\Lambda^e} M \xleftarrow{\ \partial_2\ } \cdots$$

[Barr (1966)] proves that $(\mathrm{Diff}_\Lambda(\Gamma G_K^{n+1}))_{n \geq -1}$ is a K-contractible complex of Λ^e-modules which are free relative to the underlying category of K-modules. Thus

$$H_n(\Gamma, M) \xrightarrow{\ \cong\ } \begin{cases} \mathrm{Diff}_\Lambda(\Gamma) \otimes_{\Lambda^e} M & \text{for } n = 0 \\ \mathrm{Hoch}_{n+1}(\Gamma, M) & \text{for } n > 0 \end{cases}$$

the last being Hochschild homology as defined in [Mac Lane (1963), Chapter X].

The foregoing is a K-relative homology theory for associative K-algebras, in the sense of (1.3). There is also an absolute theory, due to Shukla [Shukla (1961)], which Barr has proved coincides with the cotriple theory relative to the category of sets (with the usual dimension shift) [Barr (1967)]. We shall not deal with this absolute theory in this paper.

This concludes the present selection of examples. A further flock of examples will appear in Section 10.

2. Properties of the $H_n(X, E)_\mathbf{G}$ as functors of X, including exact sequences

Objects of the form XG, that is, values of the cotriple $\mathbf{G}$, can be thought of as **free** relative to the cotriple. Free objects are acyclic:

PROPOSITION (2.1).

$$H_0(XG, E)_\mathbf{G} \xrightarrow[\cong]{\ \lambda\ } XGE$$

$$H_n(XG, E)_\mathbf{G} = 0 \qquad \text{for } n > 0$$

An object P is called **G-projective** if P is a retract of some value of G, or equivalently, if there is a map $s : P \twoheadrightarrow PG$ such that $s \circ P\varepsilon = P$. **G**-projectives obviously have the same acyclicity property.

To prove (2.1), we just recall from the Introduction that there is a contraction in the simplicial object $(XGG^{n+1}E)_{n \geq -1}$.

If $f : X \twoheadrightarrow Y$ in $\mathbf{C}$, we define "relative groups" or homology groups of the map, $H_n(f, E)_\mathbf{G}$ for $n \geq 0$, such that the following holds:

PROPOSITION (2.2). *If $X \twoheadrightarrow Y$ in* **C***, there is an exact sequence*

$$\cdots \longrightarrow H_n(X, E)_{\mathbf{G}} \longrightarrow H_n(Y, E)_{\mathbf{G}} \longrightarrow H_n(X \twoheadrightarrow Y, E)_{\mathbf{G}}$$

$$\partial$$

$$H_{n-1}(X, E)_{\mathbf{G}} \longrightarrow \cdots \longrightarrow H_0(X \twoheadrightarrow Y, E)_{\mathbf{G}} \longrightarrow 0$$

PROPOSITION (2.3). *If $X \twoheadrightarrow Y \twoheadrightarrow Z$ in* **C***, there is an exact sequence*

$$\cdots \longrightarrow H_n(X \twoheadrightarrow Y, E)_{\mathbf{G}} \longrightarrow H_n(X \twoheadrightarrow Z, E)_{\mathbf{G}} \longrightarrow H_n(Y \twoheadrightarrow Z, E)_{\mathbf{G}}$$

$$\partial$$

$$H_{n-1}(X \twoheadrightarrow Y, E)_{\mathbf{G}} \longrightarrow \cdots \longrightarrow H_0(Y \twoheadrightarrow Z, E)_{\mathbf{G}} \longrightarrow 0$$

If 0 is an initial object in **C**, that is, if there is a unique map $0 \twoheadrightarrow X$ for every X, then 0 is **G**-projective and

$$H_0(X, E) \xrightarrow{\cong} H_0(0 \twoheadrightarrow X, E)$$

$$H_n(X, E) \xrightarrow{\cong} H_n(XG \xrightarrow{\varepsilon} X, E) \qquad \text{for } n > 0$$

Examples of these sequences will be deferred to Section 8. There we will show that under certain conditions the homology group $H_n(X \twoheadrightarrow Y, E)$ can be interpreted as a cotriple homology group relative to the natural cotriple in the category (X, C). For one thing, it will turn out that the homology of a map of commutative rings, $H_n(A \twoheadrightarrow B)$, is just the homology of B as an A-algebra.

Imitative though these sequences may be of theorems in algebraic topology, we don't know how to state a uniqueness theorem for **G**-homology in our present context.

As to the definition of the relative groups, we just let

$$H_n(X \xrightarrow{f} Y, E)_{\mathbf{G}} = H_n(Cf)$$

where Cf is the mapping cone of the chain transformation

$$fG^{n+1}E \colon XG^{n+1}E \twoheadrightarrow YG^{n+1}E \qquad \text{for } n \geq 0$$

That is,

$$(CF)_n = \begin{cases} YG^{n+1}E \oplus XG^nE & \text{for } n > 0 \\ YGE & \text{for } n = 0 \end{cases}$$

$$\partial_n = \begin{pmatrix} \partial_Y & 0 \\ fG^{n+1}E & -\partial_X \end{pmatrix} \colon (Cf)_n \twoheadrightarrow (Cf)_{n-1} \qquad \text{for } n \geq 2$$

$$\partial_1 = \begin{pmatrix} \partial_Y \\ fGE \end{pmatrix} \colon (Cf)_1 \twoheadrightarrow (Cf)_0$$

(These matrices act on row vectors from the right, ∂_X and ∂_Y indicate boundary operators in the standard complexes of X and Y.)

(2.2) follows from the exact sequence of chain complexes

$$0 \longrightarrow (Y G^{n+1} E)_{n \geq 0} \longrightarrow Cf \xrightarrow{\ \Pi\ } (X G^{n+1} E)_{n \geq 0} \longrightarrow 0$$

where the projection Π is a chain transformation of degree -1.

(2.3) follows from (2.2) by routine algebraic manipulation ([Eilenberg & Steenrod (1952), Wall (1966)]).

3. Axioms for the $H_n(X, E)_{\mathbf{G}}$ as functors of the abelian variable E

In this section we show that the functors $H_n(\ , E)_{\mathbf{G}} \colon \mathbf{C} \longrightarrow \mathscr{A}$ are characterized by the following two properties. (In Section 4 it will appear that they are characterized by a little bit less.)

3.1. $\mathbf{G}$-ACYCLICITY.

$$H_0(\ , GE)_{\mathbf{G}} \xrightarrow[\lambda]{\ \cong\ } GE,$$

$$H_n(\ , GE)_{\mathbf{G}} = 0, \qquad n > 0.$$

3.2. $\mathbf{G}$-CONNECTEDNESS. If $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ is a $\mathbf{G}$-short exact sequence of functors $\mathbf{C} \longrightarrow \mathscr{A}$, then there is a long exact sequence in homology:

$$\cdots \longrightarrow H_n(\ , E')_{\mathbf{G}} \longrightarrow H_n(\ , E)_{\mathbf{G}} \longrightarrow H_n(\ , E'')_{\mathbf{G}}$$

$$\xrightarrow{\ \partial\ }$$

$$H_{n-1}(\ , E')_{\mathbf{G}} \longrightarrow \cdots \longrightarrow H_0(\ , E'')_{\mathbf{G}} \longrightarrow 0$$

The acyclicity is trivial: as mentioned in the Introduction, the simplicial object $XG * GE$ always has a contraction by virtue of

$$GE \xrightarrow{\ \delta E\ } G(GE).$$

For the homology sequence, we define a sequence of functors $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ to be $\mathbf{G}$-exact if it is exact in the (abelian) functor category $(\mathbf{C}, \mathscr{A})$ after being composed with $G \colon \mathbf{C} \longrightarrow \mathbf{C}$, i.e., if and only if $0 \longrightarrow XGE' \longrightarrow XGE \longrightarrow XGE'' \longrightarrow 0$ is an exact sequence in $\mathscr{A}$ for every object $X \in \mathbf{C}$. In this event we get a short exact sequence of chain complexes in $\mathscr{A}$,

$$0 \longrightarrow (X G^{n+1} E') \longrightarrow (X G^{n+1} E) \longrightarrow (X G^{n+1} E'') \longrightarrow 0, \qquad n \geq -1,$$

from which the homology sequence is standard.

Next we show that properties 3.1 and 3.2 are characteristic of the homology theory $H_{\mathbf{G}}$. Define $\mathbf{L} = (L_n, \lambda, \partial)$ to be a theory of $\mathbf{G}$-left derived functors if:

1. **L** assigns to every functor $E\colon \mathbf{C} \longrightarrow \mathscr{A}$ a sequence of functors $L_n E\colon \mathbf{C} \longrightarrow \mathscr{A}$, and to every natural transformation $\vartheta\colon E \longrightarrow E_1$ a sequence of natural transformations $L_n \vartheta\colon L_n E \longrightarrow L_n E_1$, $n \geq 0$, such that $L_n(\vartheta\vartheta_1) = L_n(\vartheta) \cdot L_n(\vartheta_1)$;

2. λ is a natural transformation $L_0 E \longrightarrow E$ which has property 3.1 for every functor which is of the form GE;

3. whenever $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ is a **G**-exact sequence of functors $\mathbf{C} \longrightarrow \mathscr{A}$, then there is a long exact homology sequence

$$\cdots \longrightarrow L_n E' \longrightarrow L_n E \longrightarrow L_n E''$$
$$\xrightarrow{\ \partial\ }$$
$$L_{n-1} E' \longrightarrow \cdots \longrightarrow L_0 E'' \longrightarrow 0$$

   where ∂ actually depends on the given sequence, of course, and

$$\begin{array}{ccc} L_n E'' & \xrightarrow{\ \partial\ } & L_{n-1} E' \\ \downarrow & & \downarrow \\ L_n F'' & \xrightarrow{\ \partial\ } & L_{n-1} F' \end{array}$$

   commutes for every map of **G**-short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \end{array}$$

We now prove a uniqueness theorem for **G**-left derived functors. A proof in purely abelian-category language exists also, in fact, has existed for a long time (cf. [Röhrl (1962)] and F. Ulmer's paper in this volume.)

THEOREM (3.3). *If* **L** *is a theory of* **G**-*left derived functors, then there exists a unique family of natural isomorphisms*

$$L_n E \xrightarrow{\ \sigma_n\ } H_n(\ , E)_{\mathbf{G}}, \qquad n \geq 0,$$

which are natural in E, and are compatible with the augmentations

$$L_0 E \xrightarrow{\ \sigma_0\ } H_0(\ , E)_{\mathbf{G}}$$
$$\searrow \qquad \swarrow$$
$$E$$

and connecting homomorphisms

$$
\begin{array}{ccc}
L_n E'' & \xrightarrow{\ \partial\ } & L_{n-1}E' \\
{\scriptstyle\sigma_n}\big\downarrow & & \big\downarrow{\scriptstyle\sigma_{n-1}} \\
H_n(\ ,E'')_{\mathbf{G}} & \xrightarrow{\ \partial\ } & H_{n-1}(\ ,E')_{\mathbf{G}}
\end{array}
$$

corresponding to **G***-short exact sequences.*

PROOF. In this proof we write $H_n(\ ,E)$ for $H_n(\ ,E)_{\mathbf{G}}$.

As we shall prove in a moment, the following is a consequence of **G**-connectedness:

LEMMA (3.4). $L_0(G^2E) \xrightarrow{\ L_0\partial_1\ } L_0(GE) \xrightarrow{\ L_0\partial_0\ } L_0E \longrightarrow 0$ *is an exact sequence in the functor category* $(\mathbf{C},\mathscr{A})$.

Supposing that λ is a natural transformation $L_0E \longrightarrow E$ which is natural in E as well, we get a unique map of the cokernels

$$
\begin{array}{ccccccc}
L_0(G^2E) & \twoheadrightarrow & L_0(GE) & \longrightarrow & L_0E & \longrightarrow & 0 \\
{\scriptstyle\lambda}\big\downarrow & & {\scriptstyle\lambda}\big\downarrow & & {\scriptstyle\sigma_0}\big\downarrow & & \\
G^2E & \longrightarrow & GE & \longrightarrow & H_0(\ ,E) & \longrightarrow & 0
\end{array}
$$

which is compatible with the augmentations:

$$
\begin{array}{ccc}
L_0E & \xrightarrow{\ \sigma_0\ } & H_0(\ ,E) \\
{\scriptstyle\lambda}\searrow & & \swarrow{\scriptstyle\lambda} \\
& E &
\end{array}
$$

Now extend σ_0 inductively to a map of **G**-connected theories, $\sigma_n\colon L_nE \longrightarrow H_n(\ ,E)$ for all $n \geq 0$, as follows. Let $N = \ker(GE \longrightarrow E)$ so that

$$
0 \longrightarrow N \xrightarrow{\ i\ } GE \longrightarrow E \longrightarrow 0
$$

is an exact sequence of functors, a fortiori **G**-exact as well. As σ_0 is obviously natural in the E variable, we get a diagram

$$
\begin{array}{ccccc}
L_1E & \xrightarrow{\ \partial\ } & L_0N & \longrightarrow & L_0(GE) \\
{\scriptstyle\sigma_1}\big\downarrow & & {\scriptstyle\sigma_0}\big\downarrow & & \big\downarrow{\scriptstyle\sigma_0} \\
0 \longrightarrow H_1(\ ,E) & \xrightarrow{\ \partial\ } & H_0(\ ,N) & \longrightarrow & H_0(\ ,GE)
\end{array}
$$

the bottom row being exact by virtue of $H_1(\ ,GE) = 0$. This defines σ_1. For σ_n, $n \geq 2$, use the diagram

$$(3.6) \qquad \begin{array}{ccc} L_nE & \xrightarrow{\ \partial\ } & L_{n-1}N \\ \sigma_n \downarrow & & \downarrow \sigma_{n-1} \\ 0 \longrightarrow H_n(\ ,E) & \xrightarrow[\partial]{\cong} H_{n-1}(\ ,N) & \longrightarrow 0 \end{array}$$

This defines all of the maps σ_n. But to have a map of **G**-connected homology theories, we must verify that each square

$$\begin{array}{ccc} L_nE'' & \xrightarrow{\ \partial\ } & L_{n-1}E' \\ \sigma_n \downarrow & & \downarrow \sigma_{n-1} \\ H_n(\ ,E'') & \xrightarrow{\ \partial\ } & H_{n-1}(\ ,E') \end{array}$$

corresponding to a **G**-exact sequence $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ commutes.

We prove this first for σ_1 and σ_0, using what is basically the classical abelian-categories method. We are indebted to F. Ulmer for pointing it out. Form the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & GE & \longrightarrow & E'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & N'' & \longrightarrow & GE'' & \longrightarrow & E'' & \longrightarrow & 0 \end{array}$$

where M is $\ker(GE \longrightarrow E'')$. The left vertical arrow exists by virtue of N'' being $\ker(GE'' \longrightarrow E'')$. This induces

$$(3.7) \qquad \begin{array}{ccccc} L_1E'' & \longrightarrow & L_0M & \longrightarrow & L_0N'' \\ \sigma_1 \downarrow & & \sigma_0 \downarrow & & \downarrow \sigma_0 \\ 0 \longrightarrow H_1(\ ,E'') & \xrightarrow{\ \partial\ } & H_0(\ ,M) & \longrightarrow & H_0(\ ,N'') \end{array}$$

Since the map labeled ∂ is the kernel of $H_0(\ ,M) \longrightarrow H_0(\ ,GE)$, there exists a map $L_1E'' \longrightarrow H_1(\ ,E'')$ such that the left square commutes. As the right square commutes by naturality of σ_0, the outer rectangle commutes when this unknown map $L_1E'' \longrightarrow H_1(\ ,E'')$ is inserted. But this is exactly the property which determines σ_1

uniquely. Thus the left square commutes with σ_1 put in. As there is obviously a map

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & GE & \longrightarrow & E'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0
\end{array}
$$

a prism is induced:

$$
\begin{array}{ccc}
L_1 E'' & \longrightarrow & L_0 M \\
\end{array}
$$

The top and bottom commute by naturality of the connecting homomorphisms in the **L**- and $H(\ ,\ )$-theories, the right front face commutes by naturality of σ_0, the back commutes as it is the left square of 3.7, so the left front face also commutes, q.e.d.

The proof that the σ_n are compatible with connecting homomorphisms in dimensions > 1 is similar.

Finally, assuming that the theory **L** also satisfies the acyclicity condition 2. (or 3.1):

$$
L_0(GE) \xrightarrow[\lambda]{\cong} GE,
$$

$$
L_n(GE) = 0, \qquad n > 0,
$$

then from 3.5, $\sigma_0 \colon L_0 E \xrightarrow{\cong} H_0(\ ,E)$, and inductively from 3.7, $\sigma_n \colon L_n E \xrightarrow{\cong} H_n(\ ,E)$ for $n > 0$. This completes the uniqueness proof, except for Lemma 3.4.

Let $0 \longrightarrow N \longrightarrow GE \longrightarrow E \longrightarrow 0$ be exact, as above, and let $\nu \colon G^2 E \longrightarrow N$ be defined by the kernel property:

(3.8)

$$
\begin{array}{ccc}
 & & G^2 E \\
 & \nu \swarrow & \downarrow \partial_1 \\
0 \longrightarrow N \xrightarrow{i} & GE & \xrightarrow{\partial_0} E \longrightarrow 0
\end{array}
$$

Then

$$H_0(\ ,G^2E)$$

$$H_0(\ ,\nu) \qquad H_0(\ ,\partial_1)$$

$$H_0(\ ,N) \longrightarrow H_0(\ ,GE) \xrightarrow[H_0(\ ,\partial_0)]{} H_0(\ ,E) \longrightarrow 0$$

has an exact bottom row as $0 \longrightarrow N \longrightarrow \cdots$ is **G**-exact as well. To prove $H_0(\ ,\partial_1)$ and $H_0(\ ,\partial_0)$ exact it suffices to prove that $H_0(\ ,\nu)$ is onto. Let $K = \ker \nu$, so that $0 \longrightarrow K \longrightarrow G^2E \longrightarrow N$ is exact. Composing this with G, it is enough to show that

$$0 \longrightarrow GK \longrightarrow GG^2E \xrightarrow{G\nu} GN \longrightarrow 0$$

is exact, which just means $G\nu$ onto (in fact it turns out to be split). If we apply G to 3.8, we get

$$GG^2E$$

$$G\nu \qquad h_0 \quad G\partial_1$$

$$0 \longrightarrow GN \underset{j}{\overset{Gi}{\rightleftarrows}} GGE \underset{h_{-i}}{\overset{G\partial_0}{\rightleftarrows}} GE \longrightarrow 0$$

where the contracting maps h_{-1} and h_0 obey $G\partial_0 \cdot h_{-1} + h_0 \cdot G\partial_1 = GGE$, among other things, and the bottom row is split ($h_{-1} = \delta E$ and $h_0 = \delta GE$). Now $G\nu$ splits, for $Gi \cdot h_0 \cdot G\nu = GN$. Since Gi is a monomorphism it suffices to prove $Gi \cdot h_0 \cdot G\nu \cdot Gi = Gi$. But

$$\begin{aligned}
Gi \cdot h_0 \cdot G(\nu i) &= Gi \cdot h_0 \cdot G\partial_1 \\
&= Gi \cdot (GGE - G\partial_0 \cdot h_{-1}) \\
&= Gi - G(i\partial_0) \cdot h_{-1} \\
&= Gi
\end{aligned}$$

since $i\partial_0$ is zero. ∎

4. Homology in additive categories

Now we assume that **C** is an additive category and $\mathbf{G} = (G, \varepsilon, \delta)$ is a cotriple in **C**. It is not necessary to suppose that $G \colon \mathbf{C} \longrightarrow \mathbf{C}$ is additive or even that $0G = 0$.

If $E \colon \mathbf{C} \longrightarrow \mathscr{A}$ is a coefficient functor, and *this* is assumed to be additive, the homology functors $H_n(\ ,E)_{\mathbf{G}} \colon \mathbf{C} \longrightarrow \mathscr{A}$ are defined as before, and are additive. They admit of an axiomatic characterization like that in homological algebra (cf. 4.5).

G-projectives play a big role in the additive case. We recall $P \in \mathbf{C}$ is **G-projective** if there is a map $s\colon P \longrightarrow PG$ such that $s \cdot P\varepsilon = P$. A useful fact, holding in any category, is that the *coproduct $P * Q$ of* **G***-projectives is again* **G***-projective.*

$$P * Q \xrightarrow{\ s*t\ } PG * QG \longrightarrow (P * Q)G$$

with maps $=$, $\varepsilon * \varepsilon$, ε converging to $P * Q$.

In an additive category the coproduct is $P \oplus Q$. We assume from now on that $\mathbf{C}$ is additive.

DEFINITION 4.1.

$$X' \xrightarrow{\ i\ } X \xrightarrow{\ j\ } X''$$

is **G-exact (G-acyclic)** if $ij = 0$ and $(AG, X') \longrightarrow (AG, X) \longrightarrow (AG, X'')$ is an exact sequence of abelian groups for all $A \in \mathbf{C}$, or equivalently, if $ij = 0$ and

$$(P, X') \longrightarrow (P, X) \longrightarrow (P, X'')$$

is an exact sequence of abelian groups for every **G**-projective P. A **G-resolution** of X is a sequence $0 \longleftarrow X \longleftarrow X_0 \longleftarrow X_1 \longleftarrow \cdots$ which is **G**-acyclic and in which $X_0, X_1, \ldots$ are **G**-projective.

The usual facts about **G**-resolutions can be proved:

4.2 EXISTENCE AND COMPARISON THEOREM. **G**-resolutions always exist. If

$$0 \longleftarrow X \longleftarrow X_0 \longleftarrow X_1 \longleftarrow \cdots$$

is a **G**-projective complex and $0 \longleftarrow Y \longleftarrow Y_0 \longleftarrow Y_1 \longleftarrow \cdots$ is a **G**-acyclic complex then any $f\colon X \longrightarrow Y$ can be extended to a map of complexes

$$
\begin{array}{ccccccccc}
0 & \longleftarrow & X & \longleftarrow & X_0 & \longleftarrow & X_1 & \longleftarrow & \cdots \\
 & & \downarrow f & & \downarrow f_0 & & \downarrow f_1 & & \\
0 & \longleftarrow & Y & \longleftarrow & Y_0 & \longleftarrow & Y_1 & \longleftarrow & \cdots
\end{array}
$$

Any two such extensions are chain homotopic.

In fact,

$$0 \longleftarrow X \xleftarrow{\ \partial_0\ } XG \xleftarrow{\ \partial_1\ } XG^2 \longleftarrow \cdots$$

is a **G**-resolution of X if we let $\partial_n = \sum (-1)^i X\varepsilon_i$. It is a **G**-projective complex, and if AG is hommed into the underlying augmented simplicial object XG^*, XG^*? the resulting

simplicial set has a contraction $(AG, X)\xrightarrow{h_{-1}}(AG, XG)\xrightarrow{h_0}(AG, XG^2)\xrightarrow{h_1}\cdots$ defined by $x \cdot h_n = A\delta \cdot xG$ for $x\colon AG \longrightarrow XG^{n+1}$. Thus the simplicial group (AG, XG^*) has no homotopy, or homology, with respect to the boundary operators (AG, ∂_n).

The rest of the comparison theorem is proved just as in homological algebra.

Now we characterize the homology theory $H_n(\ , E)_\mathbf{G}\colon \mathbf{C} \longrightarrow \mathscr{A}$ by axioms on the $\mathbf{C}$ variable. In doing this we use finite projective limits in $\mathbf{C}$, although we still refrain from assuming $\mathbf{G}$ additive. We do assume that the coefficient functor $E\colon \mathbf{C} \longrightarrow \mathscr{A}$ is additive, which forces additivity of the homology functors. The axioms we get are:

4.3. $\mathbf{G}$-ACYCLICITY. If P is $\mathbf{G}$-projective, then

$$H_0(P, E)_\mathbf{G} \xrightarrow[\lambda]{\cong} PE,$$

$$H_n(P, E)_\mathbf{G} = 0, \qquad n > 0.$$

4.4. $\mathbf{G}$-CONNECTEDNESS. If $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ is a $\mathbf{G}$-exact sequence in $\mathbf{C}$, then there is a long exact sequence in homology:

$$\cdots \longrightarrow H_n(X', E)_\mathbf{G} \longrightarrow H_n(X, E)_\mathbf{G} \longrightarrow H_n(X'', E)_\mathbf{G}$$

$$\xrightarrow{\ \partial\ }$$

$$H_{n-1}(X', E)_\mathbf{G} \longrightarrow \cdots \longrightarrow H_0(X'', E)_\mathbf{G} \longrightarrow 0$$

The connecting maps are natural with respect to maps of $\mathbf{G}$-exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & 0
\end{array}
$$

It follows from 4.4 that if $X = X' \oplus X''$, then the canonical map

$$H_n(X', E)_\mathbf{G} \oplus H_n(X'', E)_\mathbf{G} \longrightarrow H_n(X' \oplus X'', E)_\mathbf{G}$$

is an isomorphism, $n \geq 0$. Thus the $H_n(\ , E)_\mathbf{G}$ are additive functors.

We are able to prove the following characterization:

4.5 UNIQUENESS. If $E_0 \xrightarrow{\lambda} E$ is a natural transformation, and $E_1, E_2, \ldots, \partial$ is a sequence of functors together with a family of connecting homomorphisms satisfying 4.3 and 4.4, then there is a unique isomorphism of connected sequences $\sigma_n\colon E_n \xrightarrow{\cong} H_n(\ , E)_\mathbf{G}$, $n \geq 0$, which commutes with the augmentations $E_0 \longrightarrow E$ and $H_0(\ , E)_\mathbf{G} \longrightarrow E$.

For the proofs of the above, $4.3 = 2.1$. For 4.4, we assume that $\mathbf{C}$ has splitting idempotents. This causes no difficulty as $\mathbf{G}$ can clearly be extended to the idempotent

completion of $\mathbf{C}$ and any abelian category valued functor can be likewise extended [Freyd (1964)]. Moreover, it is clear that this process does not affect the derived functors. (Or assume that $\mathbf{C}$ has kernels.)

Now if $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ is $\mathbf{G}$-exact it follows from exactness of

$$(X''G, X) \longrightarrow (X''G, X'') \longrightarrow 0$$

that there is a map $X''G \longrightarrow X$ whose composite with $X \longrightarrow X''$ is $X''\varepsilon$. Applying G to it we have $X''G \xrightarrow{X''\delta} X''G^2 \longrightarrow XG$ which splits $XG \longrightarrow X''G$. By our assumption we can find X_0 so that

$$0 \longrightarrow X_0 \longrightarrow XG \longrightarrow X''G \longrightarrow 0$$

is split exact. X_0 being presented as a retract of a free is $\mathbf{G}$-projective. Also, the composite

$$
\begin{array}{ccc}
X_0 & \longrightarrow & XG \\
 & & \downarrow \\
 & & \\
X & \longrightarrow & X''
\end{array}
$$

is zero and we can find $X_0 \longrightarrow X'$ so that

$$
\begin{array}{ccc}
X_0 & \longrightarrow & XG \\
\downarrow & & \downarrow \\
X' & \longrightarrow & X
\end{array}
$$

commutes. Continuing in this fashion we have a weakly split exact sequence of complexes

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_n & \longrightarrow & XG^{n+1} & \longrightarrow & X''G^{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_0 & \longrightarrow & XG & \longrightarrow & X''G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Homming a YG into it produces a weakly split exact sequence of abelian group complexes, two of which are exact, and so, by the exactness of the homology triangle, is the third. But then the first column is a **G**-projective resolution of X' and the result easily follows.

For uniqueness, 4.5, **C** must have kernels, and the argument follows the classical prescription (Section 3). This i s reasonable, for otherwise there wouldn't be enough exact sequences for 4.4 to be much of a restriction. First, $XG^2E_0 \longrightarrow XGE_0 \longrightarrow XE_0 \longrightarrow 0$ is exact in $\mathscr{A}$. Using $\lambda\colon E_0 \longrightarrow E$ one gets a unique $\sigma_0\colon E_0 \longrightarrow H_0(\ ,E)$ which is compatible with the augmentations:

$$
\begin{array}{ccc}
E_0 & \xrightarrow{\hspace{3cm}} & H_0(\ ,E) \\
& \searrow{\scriptstyle\lambda} \quad\quad {\scriptstyle\lambda}\swarrow & \\
& E &
\end{array}
$$

Letting $N = \ker(XG \longrightarrow X)$, the sequence $0 \longrightarrow N \longrightarrow XG \longrightarrow X \longrightarrow 0$ is **G**-exact. $\sigma_1\colon E_1 \longrightarrow H_1(\ ,E)$ is uniquely determined by

$$
\begin{array}{ccc}
XE_1 \longrightarrow & NE_0 \longrightarrow & XGE_0 \\
\downarrow{\scriptstyle\sigma_1} & \downarrow{\scriptstyle\sigma_0} & \downarrow{\scriptstyle\sigma_0} \\
0 \rightarrow H_1(X,E) \longrightarrow & H_0(N,E) \rightarrow & H_0(XG,E)
\end{array}
$$

σ_n similarly. Now the argument of the uniqueness part of 3.3 goes through and shows that the σ's commute with all connecting maps. Finally, if the E_n are **G**-acyclic (4.3), the σ_n are isomorphisms.

As examples we cite $\mathrm{Tor}_n^R(A,M)$ and $\mathrm{Ext}_R^n(M,A)$ obtained as **G**-derived functors, or **G**-homology, of the coefficient functors

$$ R\text{-}\mathbf{Mod} \xrightarrow{\ A\otimes_R\ } \mathbf{Ab} $$

$$ R\text{-}\mathbf{Mod}^{\mathrm{op}} \xrightarrow{\ \mathrm{Hom}_R(\ ,A)\ } \mathbf{Ab} $$

relative to the free R-module cotriple (1.1). Proved in this section are additivity of these functors and their usual axiomatic characterizations.

Similarly one gets axioms for the K-relative Tor and Ext (1.1), and for the pure Tor and Ext defined in Section 10.

4.6 APPLICATION TO SECTION 3. Let **C** be arbitrary, **G** a cotriple in **C**. Let **G** operate in the functor category $(\mathbf{C},\mathscr{A})$ by composition. The resulting cotriple is called $(\mathbf{G},\mathscr{A})$:

$$ (E)(G,\mathscr{A}) = GE \xrightarrow{\ (E)(\varepsilon,\mathscr{A})=\varepsilon E\ } E, $$

$$ (E)(G,\mathscr{A}) = GE \xrightarrow{\ (E)(\delta,\mathscr{A})=\delta E\ } G^2E = (E)(G,\mathscr{A})^2. $$

Iterating this cotriple in the usual way, we build up a simplicial functor

$$E \xleftarrow{(\varepsilon,\mathscr{A})_0} E(G,\mathscr{A}) \underset{(\varepsilon,\mathscr{A})_1}{\overset{(\varepsilon,\mathscr{A})_0}{\underset{\longleftarrow}{\overset{\longleftarrow}{\rightleftarrows}}}} E(G,\mathscr{A})^2 \; \cdots \; E(G,\mathscr{A})^{n+1} \; \cdots$$

from $\mathbf{C} \longrightarrow \mathscr{A}$. Rewritten, this is

$$E \longleftarrow GE \rightrightarrows G^2 E \; \cdots \; G^{n+1} E \; \cdots$$

Note that the i-th operator $(\varepsilon,\mathscr{A})_i \colon (E)(G,\mathscr{A})^{n+1} \longrightarrow (E)(G,\mathscr{A})^n$ is actually $\varepsilon_{n-i} E$ using the notation of the Introduction (dual spaces cause transposition). But reversing the numbering of face and degeneracy operators in a simplicial object does not change homotopy or homology. Therefore

$$H_n(E,\mathrm{id})_{(\mathbf{G},\mathscr{A})} = H_n(\ ,E)_{\mathbf{G}}, \qquad n \geq 0;$$

on the left coefficients are in the identity functor $(\mathbf{C},\mathscr{A}) \longrightarrow (\mathbf{C},\mathscr{A})$.

Thus the homology theory $H(\ ,E)_{\mathbf{G}}$ can always be obtained from a cotriple on an additive (even abelian) category, and the cotriple can be assumed additive. How can the axioms of this section be translated into axioms for the $H_n(\ ,E)_{\mathbf{G}}$ in general?

The $(\mathbf{G},\mathscr{A})$-projective functors are just the retracts of functors of the form GE. Thus the acyclicity axiom 4.3 becomes:

$$H_0(\ ,E)_{\mathbf{G}} \xrightarrow[\lambda]{\cong} E,$$

$$H_n(\ ,E)_{\mathbf{G}} = 0, \qquad n > 0,$$

if E is $(\mathbf{G},\mathscr{A})$-projective; this is equivalent to 3.1.

For the homology sequence, $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ will be $(\mathbf{G},\mathscr{A})$-exact if and only if $0 \longrightarrow GE' \longrightarrow GE \longrightarrow GE'' \longrightarrow 0$ is *split exact* in the functor category. (Prove this considering the picture

$$
\begin{array}{ccccccccc}
 & & & & & & GE'' & & \\
 & & & & & \swarrow^{s} & \downarrow & & \\
0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0
\end{array}
$$

$GE'' \xrightarrow{\delta E''} G^2 E'' \xrightarrow{Gs} GE$ splits the sequence.) $(\mathbf{G},\mathscr{A})$-exactness $\implies$ $\mathbf{G}$-exactness as defined in 3.2. The homology sequence axiom of this section is weaker than that of Section 3: it requires the exact homology sequence to be produced for a smaller class of short exact sequences.

Concepts equivalent to $(\mathbf{G},\mathscr{A})$-projectivity and -exactness have recently been employed by Mac Lane to give a projective complex $\rightarrow$ acyclic complex form to the cotriple acyclic-models comparison theorem 11.1 (unpublished). In particular, $\mathbf{G}$-representability in the acyclic-models sense (existence of $\vartheta \colon E \longrightarrow GE$ splitting the counit $\varepsilon E \colon GE \longrightarrow E$) is the same thing as $(\mathbf{G},\mathscr{A})$-projectivity, $\mathbf{G}$-contractibility is the same as $(\mathbf{G},\mathscr{A})$-acyclicity.

4.7 APPLICATION TO EXTENSIONS. Let an n-**dimensional G-extension** of X by Y be a **G**-exact sequence

$$0 \longrightarrow Y \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow X \longrightarrow 0, \qquad n > 0.$$

Under the usual Yoneda equivalence these form a set $E^n(X,Y)_{\mathbf{G}}$. $E^0(X,Y)_{\mathbf{G}} = (X,Y)$, the hom set in $\mathbf{C}$ (which is independent of $\mathbf{G}$). Using the comparison theorem 4.2, an extension gives rise to a map of complexes

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & XG^{n+1} & \longrightarrow & XG^n & \longrightarrow & \cdots & \longrightarrow & XG & \longrightarrow & X & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle a} & & \downarrow & & & & \downarrow & & \parallel{\scriptstyle =} & & \\
0 & \longrightarrow & Y & \longrightarrow & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_0 & \longrightarrow & X & \longrightarrow & 0
\end{array}
$$

The map a is an n-cocycle of X with values in the representable functor $(\ ,Y)\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$. We get in this way a map

$$E^n(X,Y)_{\mathbf{G}} \longrightarrow H^n(X,Y)_{\mathbf{G}}, \qquad n \geq 0$$

(in dimension 0, any $X \longrightarrow Y$ determines a 0-cocycle $XG \longrightarrow X \longrightarrow Y$).

In practice cotriples often have the property that $XG^2 \rightrightarrows XG \longrightarrow X$ is always a coequalizer diagram. In this case,

$$E^n(X,Y)_{\mathbf{G}} \longrightarrow H^n(X,Y)_{\mathbf{G}}$$

is an isomorphism for $n = 0$, and a monomorphism for $n > 0$. If $\mathbf{G}$ is the free cotriple in a tripleable adjoint pair $\mathbf{C} \longrightarrow \mathscr{A} \longrightarrow \mathbf{C}$ this coequalizer condition holds; in fact, in that case

$$E^n(X,Y)_{\mathbf{G}} \overset{\cong}{\longrightarrow} H^n(X,Y)_{\mathbf{G}}, \qquad n \geq 0,$$

as is proved in [Beck (1967)].

In categories of modules or abelian categories with projective generators [Huber (1962)], this gives the usual cohomological classification of extensions.

5. General notion of a **G**-resolution and the fact that the homology depends on the **G**-projectives alone

There is no shortage of resolutions from which the **G**-homology can in principle be computed, as the standard one always exists. But it would be nice to be able to choose more convenient resolutions in particular problems, and have available something like the additive comparison theorem (Section 4) in order to relate them to the standard resolutions. In fact a simplicial comparison theorem does exist [Kleisli (1967)], but we can get by with something much easier. Any category can be made freely to generate an additive category

by a well known construction and we find the solution to our problem by transferring it to this additive context. This is the same technique as is used by André [André (1967), Section 4].

The free additive category on $\mathbf{C}$, $\mathbf{ZC}$, has formal sums and differences of maps in $\mathbf{C}$ as its maps. Exact definitions and properties connected with $\mathbf{ZC}$ are given after the following definitions.

5.1 DEFINITIONS. A **G-resolution** of X is a complex

$$X \xleftarrow{\partial_0} X_0 \xleftarrow{\partial_1} X_1 \xleftarrow{} \cdots \xleftarrow{\partial_n} X_n \xleftarrow{} \cdots$$

in $\mathbf{ZC}$ in which all X_n for $n \geq 0$ are **G**-projective and which is **G**-acyclic in the sense that

$$0 \longleftarrow (AG, X)_{\mathbf{ZC}} \longleftarrow (AG, X_0)_{\mathbf{ZC}} \longleftarrow \cdots \longleftarrow (AG, X_n)_{\mathbf{ZC}} \longleftarrow \cdots$$

has zero homology in all dimensions for all values AG of the cotriple $\mathbf{G}$.

A **simplicial G-resolution** of X is an augmented simplicial object in $\mathbf{C}$

$$X \xleftarrow{\varepsilon_0} X_0 \overset{\varepsilon_0}{\underset{\varepsilon_1}{\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{\delta_0}}}}} X_1 \cdots X_n \cdots$$

$$X \xleftarrow{\partial_0=(\varepsilon_0)} X_0 \xleftarrow{\partial_1=(\varepsilon_0)-(\varepsilon_1)} X_1 \xleftarrow{} \cdots \xleftarrow{\partial_n} X_n \xleftarrow{} \cdots$$

is a **G**-resolution as defined above (as usual, $\partial_n = \sum(-1)^i(\varepsilon_i)$). In particular, the standard complex

$$X \xleftarrow{X\varepsilon_0} XG \overset{X\varepsilon_0}{\underset{X\varepsilon_1}{\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{X\delta_0}}}}} XG^2 \cdots XG^{n+1} \cdots$$

is a **G**-resolution of X, since the simplicial set $(AG, XG^{n+1})_{n \geq -1}$ has the contraction given in the proof of 4.2.

To be precise about $\mathbf{ZC}$, its objects are the same as those of $\mathbf{C}$, while a map $X \longrightarrow Y$ in $\mathbf{ZC}$ is a formal linear combination of such maps in $\mathbf{C}$, i.e., if $n_i \in \mathbf{Z}$ and $f_i \colon X \longrightarrow Y \in \mathbf{C}$, we get a map

$$X \xrightarrow{\sum n_i(f_i)} Y$$

in $\mathbf{ZC}$. (We enclose the free generators in parentheses for clarity in case $\mathbf{C}$ is already additive.) Composition is defined like multiplication in a group ring, $(\sum m_i(f_i))(\sum n_j(g_j)) = \sum\sum m_i n_j(f_i g_j)$.

The natural inclusion of the basis $\mathbf{C} \longrightarrow \mathbf{ZC}$ can be used to express the following universal mapping property. If $E \colon \mathbf{C} \longrightarrow \mathscr{A}$ is a functor into an additive category, there is a unique additive functor $\overline{E} \colon \mathbf{ZC} \longrightarrow \mathscr{A}$ such that
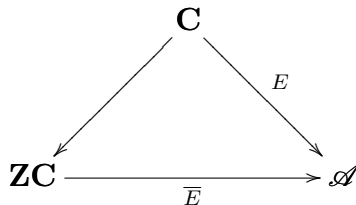
commutes. Explicitly, $X\overline{E} = XE$ and $(\sum n_i(f_i))\overline{E} = \sum n_i f_i E$.

Let $\mathbf{G} = (G, \varepsilon, \delta)$ be a cotriple in $\mathbf{C}$. Thinking of G as taking values in $\mathbf{ZC}$ we get an additive extension

$$
\begin{array}{ccc}
 & \mathbf{C} & \\
\swarrow & & \searrow^{G} \\
\mathbf{ZC} & \xrightarrow[ZG]{} & \mathbf{ZC}
\end{array}
$$

which is a cotriple $Z\mathbf{G} = (ZG, Z\varepsilon, Z\delta)$ in $\mathbf{ZC}$. Explicitly, $X \cdot ZG = XG$, and the counit and comultiplication are

$$
XG \xrightarrow{(X\varepsilon)} X,
$$

$$
XG \xrightarrow{(X\delta)} XG^2.
$$

Although there are more maps in $\mathbf{ZC}$, the notion of object does not change, and neither does the notion of projective object. For $P \in \mathbf{C}$ is $\mathbf{G}$-projective $\Leftrightarrow P$ regarded as an object in $\mathbf{ZC}$ is $Z\mathbf{G}$-projective. The forward implication is evident, and if

$$
\begin{array}{ccc}
P & \xrightarrow{\sum n_i(f_i)} & PG = P \cdot ZG \\
& \searchdownarrow & \\
\sum n_i(f_i \cdot P\varepsilon)=(P) \searrow & & \swarrow (P\varepsilon) \\
& P &
\end{array}
$$

then $f_i \cdot P\varepsilon = P$ for some i, as $(P, P)_{\mathbf{ZC}}$ is a free abelian group on a basis of which both $f_i \cdot P\varepsilon$ and P are members; this proves the other implication.

Thus the $\mathbf{G}$-resolutions of 5.1 are exactly the $Z\mathbf{G}$-resolutions relative to the cotriple in the additive category $\mathbf{ZC}$, in the sense of 4.1. Invoking the comparison theorem 4.2, we see that if (X_n) and (Y_n) are $\mathbf{G}$- or equivalently $Z\mathbf{G}$-resolutions of $X_{-1} = Y_{-1} = X$, then there is a chain equivalence

$$
(X_n) \xrightarrow{\cong} (Y_n)
$$

in $\mathbf{ZC}$.

Finally, let $E\colon \mathbf{C} \longrightarrow \mathscr{A}$ be a coefficient functor and $\overline{E}\colon \mathbf{ZC} \longrightarrow \mathscr{A}$ its additive extension constructed above. As the following complexes are identical:

$$
X(ZG)\overline{E} \xleftarrow{\partial_1} X(ZG)^2\overline{E} \longleftarrow \cdots \xleftarrow{\partial_n = (\sum_0^n (-1)^i (X\varepsilon_i))\overline{E}} X(ZG)^{n+1}\overline{E} \longleftarrow \cdots
$$

$$
XGE \xleftarrow{\partial_1} XG^2 E \longleftarrow \cdots \xleftarrow{\partial_n = \sum_0^n (-1)^i X\varepsilon_i E} XG^{n+1}E \longleftarrow \cdots
$$

we conclude that

$$
H_n(X, E)_{\mathbf{G}} = H_n(X, \overline{E})_{\mathbf{ZG}}, \qquad n \geq 0,
$$

another reduction of the general homology theory to the additive theory of Section 4. The
last equation states that the diagram

$$
\begin{array}{ccc}
& \mathbf{C} & \\
\swarrow & & \searrow \scriptstyle{H_n(\ ,E)_{\mathbf{G}}} \\
\mathbf{ZC} & \xrightarrow{\ \ H_n(\ ,\overline{E})_{Z\mathbf{G}}\ \ } & \mathscr{A}
\end{array}
$$

commutes, that is, the $H_n(\ ,\overline{E})_{\mathbf{ZG}}$ are the additive extensions of the $H_n(\ ,E)_{\mathbf{G}}$.

Parenthetically, an additive structure on $\mathbf{C}$ is equivalent to a unitary, associative
functor $\vartheta\colon \mathbf{ZC}\longrightarrow\mathbf{C}$, that is, $Z(\ )$ is a triple in the universe, and its algebras are the
additive categories; if $\mathbf{C}$ is additive, ϑ is $(\sum n_i(f_i))\vartheta = \sum n_i f_i$.

$$
\begin{array}{ccc}
\mathbf{ZC} & \xrightarrow{\ \ H_n(\ ,\overline{E})_{Z\mathbf{G}}\ \ } & \mathscr{A} \\
\scriptstyle{\vartheta}\searrow & & \swarrow \scriptstyle{H_n(\ ,E)_{\mathbf{G}}} \\
& \mathbf{C} &
\end{array}
$$

also commutes. In fact, this commutativity relation is equivalent to additivity of the
homology functors, which in turn is equivalent to the homology functors' being $Z(\ )$-
algebra maps.

In the general case—$\mathbf{C}$ arbitrary—the above gives the result that $\mathbf{G}$-homology depends
only on the $\mathbf{G}$-projectives:

THEOREM (5.2). *Let* $\mathbf{G}$ *and* $\mathbf{K}$ *be cotriples in* $\mathbf{C}$ *such that the classes of* $\mathbf{G}$*-projectives and*
$\mathbf{K}$*-projectives coincide. Then* $\mathbf{G}$ *and* $\mathbf{K}$ *determine the same homology theory, that is, there
is an isomorphism*

$$ H_n(X,E)_{\mathbf{G}} \xrightarrow{\ \cong\ } H_n(X,E)_{\mathbf{K}}, \qquad n \geq 0, $$

which is natural in both variables $X \in \mathbf{C}$, $E \in (\mathbf{C},\mathscr{A})$.

The same isomorphism holds for homology groups of a map $X \longrightarrow Y$ (see Section 2).

If $(\mathbf{G},X)$ and $(\mathbf{K},X)$ are $\mathbf{G}$ and $\mathbf{K}$ lifted to the category of objects over X, $(\mathbf{C},X)$, then
the $(\mathbf{G},X)$- and $(\mathbf{K},X)$-projectives also coincide. Thus if $E\colon (\mathbf{C},X)\longrightarrow\mathscr{A}$ is a coefficient
functor, there is an isomorphism

$$ H_n(W,E)_{(\mathbf{G},X)} \xrightarrow{\ \cong\ } H_n(W,E)_{(\mathbf{K},X)}, \qquad n \geq 0, $$

natural with respect to the variables $W \longrightarrow X \in (\mathbf{C},X)$ and $E\colon (\mathbf{C},X)\longrightarrow\mathscr{A}$.

PROOF. $P \in \mathbf{C}$ is $\mathbf{ZG}$-projective $\Leftrightarrow P$ is $Z\mathbf{K}$-projective. The augmented complexes

$$X \longleftarrow XG \longleftarrow XG^2 \longleftarrow \cdots$$
$$X \longleftarrow XK \longleftarrow XK^2 \longleftarrow \cdots$$

in $\mathbf{ZC}$ are thus projective and acyclic with respect to the same projective class in $\mathbf{ZC}$. The comparison theorem yields a chain equivalence

$$(XG^{n+1}) \overset{\cong}{\longrightarrow} (XK^{n+1})_{n \geq -1}$$

As to naturality in X, if $X \longrightarrow X_1$ in $\mathbf{C}$, the comparison theorem also says that

$$
\begin{array}{ccc}
(XG^{n+1}) & \overset{\cong}{\longrightarrow} & (XK^{n+1}) \\
\downarrow & & \downarrow \\
(X_1 G^{n+1}) & \underset{\cong}{\longrightarrow} & (X_1 K^{n+1})
\end{array}
$$

commutes up to chain homotopy. The comment about homology of a map follows from homotopy-invariance of mapping cones. $W \longrightarrow X$ being $(\mathbf{G}, X)$-projective $\Leftrightarrow W$ is $\mathbf{G}$-projective is a trivial calculation. ∎

5.2 can also be proved through the intermediary of homology in categories with models ([Appelgate (1965)], [André (1967), Section 12], and Section 10 below), as well as by a derived-functors argument (Ulmer).

To conclude this section we state the criteria for $\mathbf{G}$-resolutions and $(\mathbf{G}, X)$-resolutions which will be used in Sections 6–9.

PROPOSITION (5.3).

$$X \longleftarrow X_0 \rightleftarrows X_1 \lllrrr \cdots \lllrrr X_n \lllrrr \cdots$$

is a simplicial $\mathbf{G}$-resolution of $X = X_{-1}$ if the X_n are $\mathbf{G}$-projective for $n \geq 0$ and the following condition, which implies $\mathbf{G}$-acyclicity, holds: the cotriple $\mathbf{G}$ factors through an adjoint pair

$$
\begin{array}{ccc}
\mathbf{C} & \overset{G}{\longrightarrow} & \mathbf{C} \\
 & {}_{U}\searrow \quad \swarrow{}^{F} & \\
 & \mathbf{A} &
\end{array}
$$
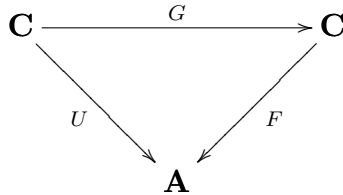
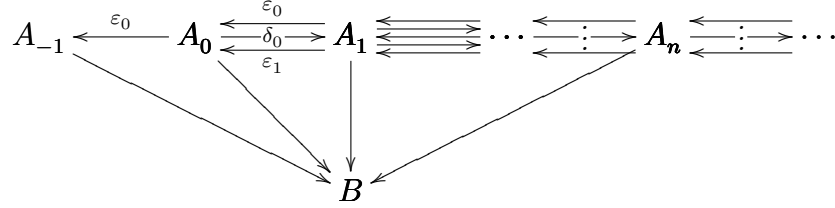and the simplicial object $(X_n U)_{n \geq -1}$ in the underlying category $\mathbf{A}$ has a contraction

$$XU \overset{h_{-1}}{\longrightarrow} X_0 U \overset{h_0}{\longrightarrow} X_1 U \overset{h_1}{\longrightarrow} \cdots \longrightarrow X_n U \overset{h_n}{\longrightarrow} \cdots$$

(satisfying $h_n \cdot \varepsilon_i U = \varepsilon_i U \cdot h_{n-1}$ for $0 \le i \le n$ and $h_n \cdot \varepsilon_{n+1} U = X_n U$). In particular, the standard **G***-resolution* $(XG^{n+1})_{n \ge -1}$ *then has such a contraction:*

$$XU \xrightarrow{\ h_{-1}\ } XGU \xrightarrow{\ h_0\ } \cdots \longrightarrow XG^{n+1}U \xrightarrow{\ h_n\ } \cdots$$

to wit, $h_n = XG^n U \eta$ *where* η *is the adjointness unit* $\eta \colon \mathbf{A} \longrightarrow FU$.

COMPLEMENT. *Let*

be a simplicial object in a category of objects over B, $(\mathbf{A}, B)$. *If*

$$A_{-1} \xrightarrow{\ h_{-1}\ } A_0 \xrightarrow{\ h_0\ } A_1 \xrightarrow{\ h_1\ } \cdots$$

is a contraction of the simplicial object sans B, *then it is also a contraction of the simplicial object in* $(\mathbf{A}, B)$, *that is, the* h_n *commute with the structural maps into* B:
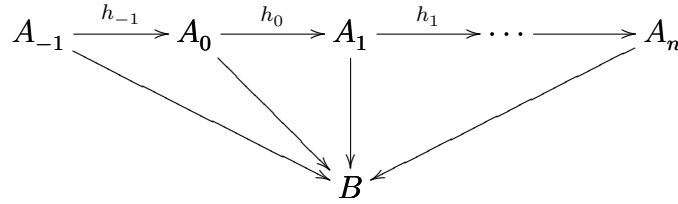
Thus when searching for contractions in categories of objects over a fixed object, the base object can be ignored.

PROOF. If the stated condition holds, the simplicial set

$$(AG, X_n)_{n \ge -1}$$

has a contraction, so the free abelian group complex $(AG, X_n)_{\mathbf{ZC}}$ has homology zero in all dimensions ≥ -1. Indeed,

$$(AG, X_n) \xrightarrow{\ k_n\ } (AG, X_{n+1}), \qquad n \ge 1,$$

is defined by

if $x\colon AG \longrightarrow X_n$.

As to the Complement, if the maps into B are $p_n\colon A_n \longrightarrow B$ then

$$h_n \cdot p_{n+1} = h_n \varepsilon_{n+1} p_n = A_n p_n = p_n$$

so in view of h_n's satisfying the identity $h_n \cdot \varepsilon_{n+1} = A_n$, it is a map over B.   ∎

6. Acyclicity and coproducts
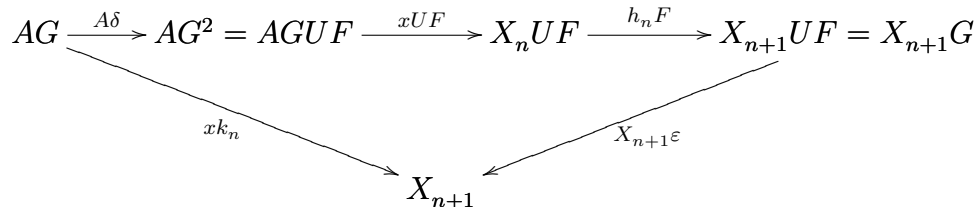
Given a **G**-resolution

$$X \;=\; X_{-1} \longleftarrow X_0 \Longleftarrow X_1 \lllessgtr \cdots$$

is its term-by-term coproduct with a fixed object Y,

$$X * Y \longleftarrow X_0 * Y \Longleftarrow X_1 * Y \lllessgtr \cdots$$

still a **G**-resolution? (The new face operators are of the form $\varepsilon_i * Y$.) If Y is **G**-projective, so are all the $X_n * Y$, $n \geq 0$. The problem is, is **G**-acyclicity preserved? In this section we consider the examples of groups, commutative algebras and (associative) algebras, and prove that acyclicity is preserved, sometimes using supplementary hypotheses. The cotriples involved come from adjoint functors

$$\mathbf{C} \xrightarrow{\;\;\;G\;\;\;} \mathbf{C}$$

with U and F maps to $\mathbf{A}$.
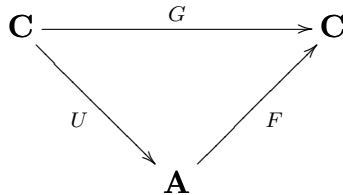
The general idea is to assume that (X_n) has a contraction in $\mathbf{A}$ and then show that this contraction somehow induces one in $(X_n * Y)$, even though the coproduct $*$ is not usually a functor on the underlying category level.

6.1. GROUPS. Let $(\Pi_n)_{n \geq -1}$ be an augmented simplicial group and $U\colon \mathscr{G} \longrightarrow \mathscr{S}$ the usual underlying set functor where $\mathscr{G}$ is the category of groups.

From simplicial topology we know that the underlying simplicial set $(\Pi_n U)$ has a contraction if and only if the natural map into the constant simplicial set

$$(\Pi_n U)_{n \geq 0} \longrightarrow (\Pi_{-1} U)$$

is a homotopy equivalence if and only if the set of components of $\Pi_* U$ is $\Pi_{-1} U$, and $H_n(\Pi_* U) = 0$ for $n > 0$ ($\Pi_* = (\Pi_n)_{n \geq 0}$). (This is because simplicial groups satisfy the Kan extension condition, hence Whitehead's Theorem; $\pi_1 = H_1$ by the group property, so the fundamental group is zero, above).

Now suppose that $(\Pi_n U)_{n \geq -1}$ is acyclic, or has a contraction, and Π is another group. We shall prove that $((\Pi_n * \Pi)\overline{U})_{n \geq -1}$ also has a contraction;

We do this by considering the group ring functor $\mathbf{Z}(\ ): \mathscr{G} \longrightarrow \mathbf{Rings}$. The simplicial ring $\mathbf{Z}\Pi_*$ obtained by applying the group ring functor in dimensionwise fashion has a contraction, namely the additive extension of the given set contraction in Π_*. In (6.3) below we shall show that the coproduct of this simplicial ring with $\mathbf{Z}\Pi$ in the category of rings, $(\mathbf{Z}\Pi_*) * \mathbf{Z}\Pi$, where the n-dimensional component is $\mathbf{Z}\Pi_n * \mathbf{Z}\Pi$, also has a contraction. But as the group ring functor is a left adjoint,

$$(\mathbf{Z}\Pi_*) * \mathbf{Z}\Pi \overset{\sim}{\longrightarrow} \mathbf{Z}(\Pi_* * \Pi)$$

Thus the set of components of the complex on the right is just $\mathbf{Z}(\Pi_{-1} * \Pi)$ and its n-th homotopy is zero for $n > 0$. This implies that $\Pi_* * \Pi$ has $\Pi_{-1} * \Pi$ as its set of components and has no higher homotopy. (This is equivalent to the curiosity that $\mathbf{Z}(\ )$ as an endofunctor on sets satisfies the hypotheses of the "precise" tripleableness theorem ([Beck (1967), Theorem 1] or [Linton (1969a)].)

6.2. COMMUTATIVE ALGEBRAS. First let $\mathbf{G}_A$ be the cotriple relative to A-modules:

$$(A, \mathbf{Comm}) \xrightarrow{\ G_A\ } (A, \mathbf{Comm})$$

with U down-right to A-$\mathbf{Mod}$ and F up-right.

$\mathbf{G}_A$-resolutions behave very well with respect to coproducts of commutative A-algebras, $B \otimes_A C$. Indeed, as the standard resolution

$$B \longleftarrow BG_A \Lleftarrow BG_A^2 \lllllll \cdots$$

has an A-*linear* contraction (5.3), so has

$$B \otimes_A C \longleftarrow BG_A \otimes_A C \Lleftarrow BG_A^2 \otimes_A C \lllllll \cdots .$$

On the other hand, let $\mathbf{G}$ be the absolute cotriple

$$K\text{-}\mathbf{Alg} \xrightarrow{\ G\ } K\text{-}\mathbf{Alg}$$

with U down-right to K-$\mathbf{Mod}$ and F up-right.

The standard $\mathbf{G}$-resolution has a contraction on the underlying set level (5.3). Thus the chain complex of A-modules associated to

$$B \longleftarrow BG \Lleftarrow BG^2 \lllllll \cdots$$

is an A-free resolution of B as an A-module in the usual homological sense. Thus the nonnegative-dimensional part of

$$B \otimes_A C \longleftarrow BG \otimes_A C \Longleftarrow BG^2 \otimes_A C \Lleftarrow \cdots$$

has $H_n = \mathrm{Tor}_n^A(B, C)$, $n \geq 0$. Since it is also a group complex, this simplicial object will have a contraction as a simplicial set $\Leftrightarrow \mathrm{Tor}_n^A(B, C) = 0$, $n > 0$.

6.3. RESOLUTIONS AND COPRODUCTS OF ASSOCIATIVE ALGEBRAS. Let K-**Alg** be the category of associative K-algebras with identity. We are interested in resolutions relative to the adjoint pair

$$K\text{-}\mathbf{Alg} \xrightarrow{\quad G \quad} K\text{-}\mathbf{Alg}$$
$$U \searrow \qquad \nearrow F$$
$$K\text{-}\mathbf{Mod}$$
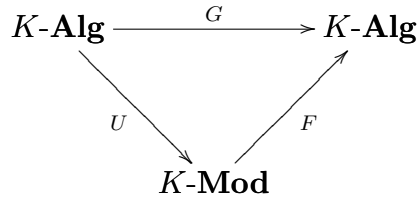
These will give rise to Hochschild homology. Here F is the K-tensor algebra $MF = K + M + M \otimes M + M \otimes M \otimes M + \cdots$. If Λ, Γ are K-algebras, their coproduct

$$\Lambda * \Gamma = (\Lambda + \Gamma)F/I$$

where I is the 2-sided ideal generated by the elements

$$\lambda_1 \otimes \lambda_2 - \lambda_1\lambda_2,\ \gamma_1 \otimes \gamma_2 - \gamma_1\gamma_2,\ 1_K - 1_\Lambda,\ 1_K - 1_\Gamma$$

(1_K is in the summand of degree 0, 1_Λ and 1_Γ are in the summand of degree 1). The K-linear maps $\Lambda, \Gamma \longrightarrow (\Lambda \oplus \Gamma)F$ become algebra maps when I is divided out and these two maps are the coproduct injections $\Lambda, \Gamma \longrightarrow \Lambda * \Gamma$. (In fact, I is the smallest ideal which makes these maps of unitary K-algebras.)

Let $(\Lambda_n)_{n \geq -1}$ be an augmented simplicial algebra which is U-contractible, i.e., there exists a K-linear contraction

$$\Lambda_{-1}U \xrightarrow{\ h_{-1}\ } \Lambda_0 U \xrightarrow{\ h_0\ } \Lambda_1 U \longrightarrow \cdots$$

We want to know that such a contraction continues to exist in the simplicial algebra $(\Lambda_n * \Gamma)_{n \geq -1}$. But we can only prove this in a special case.

An algebra Λ is called K-*supplemented* if there is a K-linear map $\Lambda \longrightarrow K$ such that $K \longrightarrow \Lambda \longrightarrow K$ is the identity of K. (The first map sends $1_K \longrightarrow 1_\Lambda$). An algebra map $\Lambda \longrightarrow \Lambda_1$ is called K-*supplemented* if Λ, Λ_1 are K-supplemented and

$$\Lambda \longrightarrow \Lambda_1$$
$$\searrow \qquad \swarrow$$
$$K$$
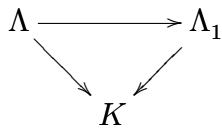
commutes.

We will show that if Λ, Γ are K-supplemented, then the coproduct of the canonical resolution of Λ with Γ,

$$\Lambda * \Gamma \longleftarrow \Lambda G * \Gamma \Longleftarrow \Lambda G^2 * \Gamma \Lllearrow \cdots$$

possesses a K-contraction. We refer to (5.3) for the fact that $(\Lambda G^{n+1})_{n \geq 1}$ always has a K-linear contraction, and we prove that this contraction survives into the coproduct of the resolution with Γ.

The cotriple $\mathbf{G}$ operates in a natural way in the category of K-supplemented algebras. For if Λ is K-supplemented, the composition $\Lambda G \longrightarrow \Lambda \longrightarrow K$ defines a K-supplementation of ΛG. If $\Lambda \longrightarrow \Lambda_1$ is K-supplemented, so is the induced $\Lambda G \longrightarrow \Lambda_1 G$,

$$
\begin{array}{ccc}
\Lambda G & \longrightarrow & \Lambda_1 G \\
\downarrow & & \downarrow \\
\Lambda & \longrightarrow & \Lambda_1 \\
& \searrow \quad \swarrow & \\
& K &
\end{array}
$$

and if Λ is K-supplemented, the counit and comultiplication maps $\Lambda G \longrightarrow \Lambda$, $\Lambda G \longrightarrow \Lambda G^2$ are also K-supplemented.

When Λ is K-supplemented let $\overline{\Lambda} = \ker(\Lambda \longrightarrow K)$. If $f \colon \Lambda \longrightarrow \Lambda_1$ is K-supplemented, then $f = K \oplus \overline{f}$ where $\overline{f} \colon \overline{\Lambda} \longrightarrow \overline{\Lambda}_1$ is induced in the obvious way. This means that if we write $f \colon K \oplus \overline{\Lambda} \longrightarrow K \oplus \overline{\Gamma}$ in the form of 2×2 matrix, the matrix is diagonal:

$$
\begin{pmatrix} K & 0 \\ 0 & \overline{f} \end{pmatrix}
$$

Using the above supplementation and writing $\Lambda G^{n+1} = K \oplus \overline{\Lambda G^{n+1}}$, all of the face operators in the standard resolution $(\Lambda G^{n+1})_{n \geq -1}$ will be diagonal:

$$K \oplus \overline{\Lambda G^n} \xleftarrow{\begin{pmatrix} K & 0 \\ 0 & \overline{\varepsilon}_i \end{pmatrix}} K \oplus \overline{\Lambda G^{n+1}}, \quad 0 \leq i \leq n \ .$$

The K-linear contraction $h_n \colon \Lambda G^{n+1} \longrightarrow \Lambda G^{n+2}$ is given by a 2×2 matrix

$$K \oplus \overline{\Lambda G^{n+1}} \xrightarrow{\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}} K \oplus \overline{\Lambda G^{n+2}}$$

The relation $h_n \varepsilon_{n+1} = \Lambda G^{n+1}$ is equivalent to

$$h_{11} = K, \quad h_{21} = 0, \quad h_{12}\overline{\varepsilon}_{n+1} = 0, \quad h_{22}\overline{\varepsilon}_{n+1} = \overline{\Lambda G^{n+1}}$$

(the matrix acts on row vectors from the right). The relation $h_n \varepsilon_i = \varepsilon_i h_n$, $0 \le i \le n$, is equivalent to $h_{22} \bar{\varepsilon}_i = \bar{\varepsilon}_i h_{22}$ as another matrix calculation shows. Thus the contraction matrix has the form

$$\begin{pmatrix} K & h_{12} \\ 0 & h_{22} \end{pmatrix}$$

where entry h_{22} satisfies the contraction identities with respect to the restrictions of the face operators ε_i to the supplementation kernels, i.e. with respect to the maps $\bar{\varepsilon}_i$. Thus, we can switch to

$$h'_n = \begin{pmatrix} K & 0 \\ 0 & h_{22} \end{pmatrix}$$

which is also a matrix representation of a K-linear contraction

$$\Lambda G'^{n+1} \xrightarrow{h'_n} \Lambda G'^{n+2}$$

$$K \oplus \overline{\Lambda G^{n+1}} \xrightarrow{K \oplus \overline{h_n}} K \oplus \overline{\Lambda G^{n+2}}$$

where we have written $\bar{h}_n$ in place of h_{22}. This change can be made for all $n \ge -1$, so we get a K-contraction which is in diagonal form.

The next step is to find that the coproduct of two K-supplemented algebras can be written in a special form. Consider the direct sum

$$W = K + \overline{\Lambda} + \overline{\Gamma} + \overline{\Lambda} \otimes \overline{\Gamma} + \overline{\Gamma} \otimes \overline{\Lambda} + \overline{\Lambda} \otimes \overline{\Gamma} \otimes \overline{\Lambda} + \cdots$$

of all words formed by tensoring $\overline{\Lambda}$ and $\overline{\Gamma}$ together with no repetitions allowed. There is an evident K-linear map

$$W \longrightarrow \Lambda * \Gamma$$

given on the fifth summand above, for example, by

$$\overline{\Lambda} \otimes \overline{\Gamma} \otimes \overline{\Lambda} \longrightarrow \Lambda \otimes \Gamma \otimes \Lambda \longrightarrow (\Lambda \oplus \Gamma)F \longrightarrow \Lambda * \Gamma$$

The map $W \longrightarrow \Lambda * \Gamma$ is one-one because its image in F does not intersect the ideal I $((\Lambda + \Gamma)F/I = \Lambda * \Gamma)$ and it is onto, clearly. Thus viewing $\Lambda * \Gamma$ as a K-module, we have

$$\Lambda * \Gamma = K + \overline{\Lambda} + \overline{\Gamma} + \overline{\Lambda} \otimes \overline{\Gamma} + \cdots$$

Now let $\Lambda \xrightarrow{f} \Lambda_1$, $\Gamma \xrightarrow{g} \Gamma_1$, be K-*linear* maps which respect both units and supplementations, that is, writing $\Lambda = K \oplus \overline{\Lambda}$, and $\Lambda_1, \Gamma, \Gamma_1$ similarly,

$$K \oplus \overline{\Lambda} \xrightarrow{f = K \oplus \overline{f}} K \oplus \overline{\Lambda}_1$$

$$K \oplus \overline{\Gamma} \xrightarrow{g = K \oplus \overline{g}} K \oplus \overline{\Gamma}_1$$

Then a K-linear map $\Lambda * \Gamma \longrightarrow \Lambda_1 * \Gamma_1$, which we take the liberty of denoting by $f * g$, is induced:

$$K + \overline{\Lambda} + \overline{\Gamma} + \overline{\Lambda} \otimes \overline{\Gamma} + \cdots$$
$$\downarrow \overline{f} \qquad \downarrow \overline{g} \qquad \downarrow \overline{f} \otimes \overline{g}$$
$$K + \overline{\Lambda}_1 + \overline{\Gamma}_1 + \overline{\Lambda}_1 \otimes \overline{\Gamma}_1 + \cdots$$

If we are also given $f_1 \colon \Lambda_1 \longrightarrow \Lambda_2$, $g_1 \colon \Gamma_1 \longrightarrow \Gamma_2$, then $f_1 * g_1 \colon \Lambda_1 * \Gamma_1 \longrightarrow \Lambda_2 * \Gamma_2$ and functoriality holds: $(f * g)(f_1 * g_1) = (f f_1 * g g_1)$ (because $\overline{f f_1} = \overline{f} \, \overline{f}_1$, and similarly for g, g_1).

To complete the argument, let

$$\Lambda \xrightarrow{\ K \oplus \overline{h}_{-1}\ } \Lambda G \xrightarrow{\ K \oplus \overline{h}_0\ } \Lambda G^2 \longrightarrow \cdots \longrightarrow \Lambda G^{n+1} \xrightarrow{\ K \oplus \overline{h}_n\ } \cdots$$

be the K-contraction with diagonal matrix constructed above. The $K \oplus h_n$ preserve both units and supplementations, so

$$\Lambda * \Gamma \xrightarrow{\ (K \oplus \overline{h}_{-1}) * \Gamma\ } \Lambda G * \Gamma \longrightarrow \cdots \longrightarrow \Lambda G^{n+1} * \Gamma \xrightarrow{\ (K \oplus \overline{h}_n) * \Gamma\ } \cdots$$

is a sequence of well defined K-linear maps which satisfies the contraction identities by virtue of the above functoriality.

ALTERNATIVE ARGUMENT. The commutative diagram of adjoint pairs
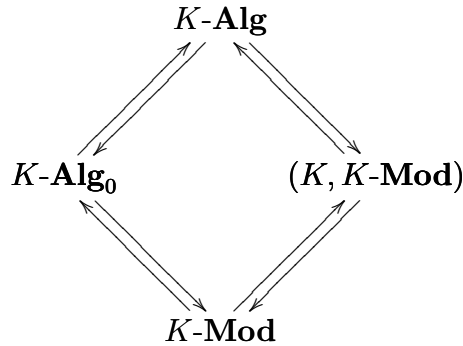
arises from a distributive law $TS \longrightarrow ST$ in K-**Mod**; T is the triple $M \longrightarrow K \oplus M$, whose algebras are *unitary* K-modules (objects of the comma category $(K, K\text{-}\mathbf{Mod})$) and S is the triple $M \longrightarrow M + M \otimes M + \cdots$, whose algebras are associative K-algebras without unit; this is the category denoted $K\mathbf{Alg}_0$. Let $\mathbf{G}$ be the cotriple in K-**Alg** relative to K-**Mod**, and $\mathbf{G}_1$ that relative to unitary K-modules $(K, K\text{-}\mathbf{Mod})$. By an easy extrapolation of [Barr (1969), 5.2], the cotriples $\mathbf{G}$, $\mathbf{G}_1$ operate in the full subcategory K-$\mathbf{Alg}'$ consisting of those $\Lambda \in K$-**Alg** whose underlying unitary K-modules are projective relative to K-modules, and $\mathbf{G}$, $\mathbf{G}_1$ restricted to this subcategory have the same projective objects. Now Λ, as a unitary K-module, is projective relative to K-modules $\Longleftrightarrow$ there is a commutative

diagram of K-linear maps

$$\begin{array}{c} K \\ \swarrow \quad \downarrow \quad \searrow \\ \Lambda \longrightarrow K \oplus \Lambda \longrightarrow \Lambda \end{array}$$

$\iff \Lambda$ has a K-linear supplementation. Thus if Λ is such an algebra, the standard resolutions

$$(\Lambda G^{n+1}), \ (\Lambda G_1^{n+1})_{n \geq -1}$$

are chain equivalent in $\mathbf{Z}(K\text{-}\mathbf{Alg})$. $(\ ) * \Gamma$ extends to an additive endofunctor of $\mathbf{Z}(K\text{-}\mathbf{Alg})$.

$$(\Lambda G^{n+1} * \Gamma), \ (\Lambda G_1^{n+1} * \Gamma)_{n \geq -1}$$

are therefore also chain equivalent in $\mathbf{Z}(K\text{-}\mathbf{Alg})$. Finally, $(\Lambda G_1^{n+1} * \Gamma)_{n \geq -1}$ has a K-linear contraction. This implies that

$$H_p(\Lambda G, (\Lambda G^{n+1} * \Gamma)_{n \geq -1}) = 0, \qquad p \geq 0,$$

so $(\Lambda G^{n+1} * \Gamma)$ is $\mathbf{G}$-acyclic.

As to the last K-contraction, if Λ is any K-algebra for a moment, and Γ is K-linearly supplemented, then as a K-module $\Lambda * \Gamma$ can be viewed as a direct sum

$$\Lambda + \Gamma + \Lambda \otimes \overline{\Gamma} + \overline{\Gamma} \otimes \Lambda + \cdots$$

modulo the relations $\gamma \otimes 1_\Lambda = 1_\Lambda \otimes \gamma = \gamma$, and the ideal generated by them, such as $\lambda \otimes \gamma \otimes 1_\Lambda = \lambda \otimes \gamma, \ldots$. Thus if $f \colon \Lambda \longrightarrow \Lambda_1$ is a *unitary* K-linear map, "$f * \Gamma$": $\Lambda * \Gamma \longrightarrow \Lambda_1 * \Gamma$ is induced, and functoriality holds: $f f_1 * \Gamma = (f * \Gamma)(f_1 * \Gamma)$. Now the resolution $(\Lambda G_1^{n+1})_{n \geq -1}$ has a *unitary* K-linear contraction (5.3). This contraction goes over into $(\Lambda G_1^{n+1} * \Gamma)_{n \geq -1}$ provided Γ is K-linearly supplemented.

7. Homology coproduct theorems

Let $\mathbf{G}$ be a cotriple in $\mathbf{C}$ and let E be a coefficient functor $\mathbf{C} \longrightarrow \mathscr{A}$. *E preserves coproducts* if the map induced by the coproduct injections $X, Y \longrightarrow X * Y$ is an isomorphism for all $X, Y \in \mathbf{C}$:

$$XE \oplus YE \xrightarrow{\ \cong\ } (X * Y)E$$

(In this section we assume $\mathbf{C}$ has coproducts). Particularly if E preserves coproducts, it is plausible that the similarly-defined natural map in homology is an isomorphism; if it is indeed the case that

$$H_n(X, E)_{\mathbf{G}} \oplus H_n(Y, E)_{\mathbf{G}} \longrightarrow H_n(X * Y, E)_{\mathbf{G}}$$

is an isomorphism, we say that *the homology coproduct theorem holds* (strictly speaking, for the objects X, Y, in dimension n; it is characteristic of the theory to be developed that the coproduct theorem often holds only for objects X, Y with special properties).

In this section we show that the homology coproduct theorem holds for the various categories and various cotriples considered in Section 6. As one gathers from the arguments resorted to in that section, there must be something the matter with the slick method of proving coproduct theorems sketched in [Barr & Beck (1966), §5]. First, to correct a slip, (5.4) in [Barr & Beck (1966)] should read $u\colon (X_1 * X_2)GU \longrightarrow (X_1 G * X_2 G)U$, that is, it is in the underlying category that u should be sought. However, even with that correction, such a natural u does not exist so far as we know, in group theory (relative to sets) or in Hochschild theory, contrary to our earlier claims. The morphisms u which we had in mind in these cases turned out on closer inspection not to be natural, because of misbehavior of neutral elements of one kind or another in coproducts viewed at the underlying-category level. Only in "case 3" of [Barr & Beck (1966), §5], namely that of commutative algebras relative to K-modules, does the method of that paper work. However, we are able to retrieve most of the results claimed there although in the case of Hochschild theory we are forced to impose an additional linear-supplementation hypothesis.

Such tests as we possess for the coproduct theorem are contained in the next two propositions.

PROPOSITION (7.1). *If X and Y possess $\mathbf{G}$-resolutions*

$$X \ = \ X_{-1} \longleftarrow X_0 \Longleftarrow X_1 \lllllll \cdots$$

$$Y \ = \ Y_{-1} \longleftarrow Y_0 \Longleftarrow Y_1 \lllllll \cdots$$

such that the coproduct

$$X * Y \longleftarrow X_0 * Y_0 \Longleftarrow X_1 * Y_1 \lllllll \cdots$$

is a $\mathbf{G}$-resolution, then the coproduct theorem holds for X, Y and any coproduct-preserving coefficient functor. (The issue is $\mathbf{G}$-acyclicity.)

In particular, if each row and column of the double augmented simplicial object

$$(X_m * Y_n)_{m,n \geq -1}$$

is a $\mathbf{G}$-resolution, then the above diagonal object $(X_n * Y_n)_{n \geq -1}$ is a $\mathbf{G}$-resolution.

PROPOSITION (7.2). *Suppose that the cotriple $\mathbf{G}$ factors through an adjoint pair*
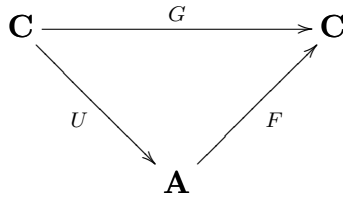
and that $(X * Y)U$ is naturally equivalent to $XU *_{\mathbf{A}} YU$ where $*_{\mathbf{A}}\colon \mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{A}$ is some bifunctor; in other words, the coproduct is definable at the underlying-category level. Then the homology coproduct theorem holds for any coproduct-preserving coefficient functor.

As to (7.1), coproducts of projectives being projective, we are left to consider the augmented double simplicial set

$$(AG, X_m * Y_n)_{m,n \geq -1}$$

As the rows and columns lack homology, so does the diagonal, by the Eilenberg-Zilber theorem [Eilenberg & Zilber (1953)]. For (7.2), identify

$$((XG^{n+1} * YG^{n+1})U)_{n \geq -1} \text{ with } (XG^{n+1}U *_{\mathbf{A}} YG^{n+1}U),$$

of which

$$XU *_{\mathbf{A}} \xrightarrow{h_{-1}*_{\mathbf{A}}k_{-1}} XGU *_{\mathbf{A}} YGU \longrightarrow \cdots$$

is a contraction (see (5.3)).

In the following examples we use the fact that the coproduct in the category of objects over X, $(\mathbf{C}, X)$, is "the same" as the coproduct in $\mathbf{C}$:

$$(X_1 \xrightarrow{P_1} X) * (X_2 \xrightarrow{P_1} X) = (X_1 * X_2 \xrightarrow{(P_1, P_2)} X)$$

7.3. GROUPS.

$$H_n(\Pi_1, E) \oplus H_n(\Pi_2, E) \xrightarrow{\sim} H_n(\Pi_1 * \Pi_2, E)$$

for any coproduct-preserving functor $E: G \longrightarrow \mathscr{A}$, such as $\otimes M$ or $\mathrm{Hom}(\ , M)$ where M is a fixed abelian group.

To deduce the usual coproduct theorems for homology and cohomology with coefficients in a module, we apply the complement to (5.3) to see that $(\Pi_1 G^{n+1})_{n \geq -1}$ is a $(\mathbf{G}, \Pi_1 * \Pi_2)$-resolution of Π_1 as a group over $\Pi_1 * \Pi_2$ (using the coproduct injection $\Pi_1 \longrightarrow \Pi_1 * \Pi_2$). By (6.1) and the complement to (5.3) again, $(\Pi_1 G^{n+1} * \Pi_2 G^{n+1})_{n \geq -1}$ is a $(\mathbf{G}, \Pi_1 * \Pi_2)$-resolution of $\Pi_1 * \Pi_2 \longrightarrow \Pi_1 * \Pi_2$. If M is a $\Pi_1 * \Pi_2$-module, then M can be regarded both as a Π_1-module and as a Π_2-module by means of $\Pi_1, \Pi_2 \longrightarrow \Pi_1 * \Pi_2$. Thus in homology we have a chain equivalence between the complexes

$$(\mathrm{Diff}_{\Pi_1 * \Pi_2}(\Pi_1 G^{n+1}) \otimes M \oplus \mathrm{Diff}_{\Pi_1 * \Pi_2}(\Pi_2 G^{n+1}) \otimes M),$$
$$(\mathrm{Diff}_{\Pi_1 * \Pi_2}(\Pi_1 G^{n+1} * \Pi_2 G^{n+1}) \otimes M), \qquad\qquad n \geq 0,$$

$\otimes$ being over $\Pi_1 * \Pi_2$. As a result,

$$H_n(\Pi_1, M) \oplus H_n(\Pi_2, M) \xrightarrow{s} im H_n(\Pi_1 * \Pi_2, M).$$

In cohomology, taking coefficients in $\mathrm{Hom}_{\Pi_1 * \Pi_2}(\ , M)$,

$$H^n(\Pi_1 * \Pi_2, M) \xrightarrow{\sim} H^n(\Pi_1, M) \oplus H^n(\Pi_2, M).$$

These isomorphisms, apparently known for some time, appear to have been first proved (correctly) in print in [Barr & Rinehart (1966)]. Similar isomorphisms hold for (co-)homology of $W_1 * W_2$ where $W_1 \longrightarrow \Pi_1$, $W_2 \longrightarrow \Pi_2$ are groups over Π_1, Π_2. (Earlier proof: [Trotter (1962)].)

7.4. COMMUTATIVE ALGEBRAS. Let B, C be A-algebras over D, that is, $B, C \longrightarrow D$, and M a D-module.

If H is homology relative to the "absolute" cotriple $\mathbf{G}$ coming from $(A, \mathbf{Comm}) \longrightarrow \mathbf{Sets}$, we have

$$H_n(B, M) \oplus H_n(C, M) \overset{\sim}{\longrightarrow} H_n(B \otimes_A C, M),$$

$$H^n(B \otimes_A C, M) \overset{\sim}{\longrightarrow} H^n(B, M) \oplus H^n(C, M)$$

provided $\mathrm{Tor}_p^A(B, C) = 0$ for $p > 0$; this is because the coproduct of the standard resolutions,

$$(BG^{n+1} \otimes_A CG^{n+1})_{n \geq -1},$$

has $\mathrm{Tor}(B, C)$ as its homology (use the Eilenberg-Zilber theorem), which is the obstruction to a contraction in the underlying category of sets. The result is also proved in [André (1967), Quillen (1967), Lichtenbaum & Schlessinger (1967), Harrison (1962)].

If H is the theory relative to A-modules, the isomorphisms hold without any condition (6.2).

7.5. ASSOCIATIVE K-ALGEBRAS. If $\Lambda_1, \Lambda_2 \longrightarrow \Gamma$ are K-algebra maps and M is a two-sided Γ-module, and Λ_1, Λ_2 possess K-linear supplementations, then

$$H_n(\Lambda_1, M) \oplus H_n(\Lambda_2, M) \overset{\sim}{\longrightarrow} H_n(\Lambda_1 * \Lambda_2, M),$$

$n \geq 0$; the same cohomology with coefficients in M, or for any coproduct-preserving coefficient functor. The cotriple employed is that relative to K-modules; the proofs are from (6.3), (7.1).

8. On the homology of a map

In Section 2 we defined homology groups of a map so as to obtain an exact sequence

$$\cdots \longrightarrow H_n(X, E) \longrightarrow H_n(Y, E) \longrightarrow H_n(X \longrightarrow Y, E) \overset{\partial}{\longrightarrow} H_{n-1}(X, E) \longrightarrow \cdots$$

In fact, although we had to use a mapping cone instead of a quotient complex, the definition is the same as in algebraic topology. In this section we show (with a proviso) that these groups are the same as the cotriple groups

$$H_n(X \longrightarrow Y, (X, E))_{(X, \mathbf{G})}, \qquad n \geq 0,$$

where $X \longrightarrow Y$ is considered as an object under X, (X, E) is the extension to a functor $(X, \mathbf{C}) \longrightarrow \mathscr{A}$ of a given coefficient functor $E \colon \mathbf{C} \longrightarrow \mathscr{A}$ and $(X, \mathbf{G})$ is $\mathbf{G}$ lifted into the comma category as described before (1.2); the *proviso* is that a homology coproduct theorem should hold for the coproduct of any object with a free object.

The coefficient functor we use,

$$(X, \mathbf{C}) \xrightarrow{\;(X, E)\;} \mathscr{A},$$

is defined by $(X \longrightarrow Y)(X, E) = \operatorname{coker} XE \longrightarrow YE$. Recalling the formulas for $(X, \mathbf{G})$, we have that the $H_n(X \longrightarrow Y, (X, E))_{X,\mathbf{G}}$ are the homology groups of the standard complex which in dimensions 0 and 1 reads:

$$0 \leftarrow \operatorname{coker} (XE \longrightarrow (X * YG)E) \xleftarrow{\ \partial_1\ } \operatorname{coker} (XE \longrightarrow (X * (X * YG)G)E) \xleftarrow{\ \partial_2\ } \cdots.$$

THEOREM (8.1). *There is a sequence of homology maps*

$$H_n(X \longrightarrow Y, E)_{\mathbf{G}} \xrightarrow{\ H_n(\varphi)\ } H_n(X \longrightarrow Y, (X, E))_{(X,\mathbf{G})}, \qquad n \geq 0,$$

resulting from a natural chain transformation

$$C(X \longrightarrow Y)_n \xrightarrow{\ \varphi_n\ } (X \longrightarrow Y)(X, \mathbf{G})^{n+1}(X, E).$$

($C(X \longrightarrow Y)$ is the mapping defined in Section 2 and functoriality is respect to maps of objects under X). The $H_n(\varphi)$ are isomorphisms *if the following theorem holds: for all $X, Y \in \mathbf{C}$, the coproduct injections induce isomorphisms*

$$H_n(X * YG, E)_{\mathbf{G}} \xleftarrow{\ \sim\ } \begin{cases} H_0(X, E)_{\mathbf{G}} \oplus YGE, & n = 0, \\ H_n(X, E)_{\mathbf{G}}, & n > 0, \end{cases}$$

that is, if E satisfies the homology coproduct theorem when one summand is $\mathbf{G}$-free.

PROOF. We augment both complexes by attaching H_0 as (-1)-dimensional term. We first define φ_0, φ_1 so as to obtain the commutative square $\varphi_1 \partial_1 = \partial_1 \varphi_0$, which induces a natural map φ_{-1} on the augmentation terms.

$$
\begin{array}{ccccc}
YG^2E \oplus XGE & \xrightarrow{\ \varphi_1\ } & (X * (X * YG)G)E & \xrightarrow{\ \operatorname{coker}\ } & (X \rightarrow Y)(X, G)^2(X, E) \\
\downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_1} \\
YGE & \xrightarrow[\ \varphi_0\ ]{} & (X * YG)E & \xrightarrow[\ \operatorname{coker}\ ]{} & (X \rightarrow Y)(X, G)(X, E) \\
\downarrow & & \downarrow & & \downarrow \\
H_0(X \rightarrow Y, E)_{\mathbf{G}} & & \xrightarrow[\ \varphi_{-1}\ ]{} & & H_0(X \rightarrow Y, (X, E))_{(X,\mathbf{G})}
\end{array}
$$

If we write $i \colon X \longrightarrow Y * YG$, $j \colon YG \longrightarrow X * YG$ for coproduct injections, then $\varphi_0 = jE$, and φ_1 is determined by

$$
\begin{array}{c}
XGE \\
\searrow{\scriptstyle iGE} \\
(X * YG)GE \xrightarrow{\ j_1E\ } (X * (X * YG)G)E \\
\nearrow{\scriptstyle jGE} \\
YG^2E
\end{array}
$$

where j_1 is also a coproduct injection. That φ_0, φ_1 commute with ∂_1 is readily checked.

The higher φ_n could be written down similarly but we don't bother with that as they automatically fall out of the acyclic-models argument which we need for the isomorphism anyway. We use (X, G) as the comparison cotriple. The cotriple complex $(X \longrightarrow Y)(X, G)^{n+1}(X, E)$ is representable and contractible with respect to this cotriple, as always. Furthermore, $C(X \longrightarrow Y)$ is (X, G)-representable via

$$\vartheta_n \colon C(X \longrightarrow Y)_n \longrightarrow C(X \longrightarrow X * YG)_n$$

$$YG^{n+1}E \oplus XG^nE \xrightarrow{(Y\delta.jG)G^nE\oplus \mathrm{id.}} (X * YG)G^{n+1}E \oplus XG^nE,$$

if $n > 0$, and $\vartheta_0 = jE$. This proves φ_{-1} can be extended to a chain transformation defined in all dimensions. It happens that the extension produced by (11.1) agrees with the above φ_0, φ_1 in the lowest dimensions.

To conclude, if the homology coproduct assumption in (8.1) holds, then

$$H_n(X \longrightarrow X * YG, E)_{\mathsf{G}} \;\simeq\; \begin{cases} YGE, & n = 0, \\ 0, & n > 0, \end{cases}$$

since this homology group H_n fits into the exact sequence

$$\cdots \longrightarrow H_n(X, E) \longrightarrow H_n(X * YG, E) \longrightarrow H_n$$

$$\partial \cdots\cdots$$

$$H_0(X, E) \longrightarrow H_0(X * YG, E) \longrightarrow H_0 \longrightarrow 0$$

Thus the φ_n induce homology isomorphisms between the two theories (11.3). ∎

8.2. GROUPS. If

$$\Pi_0 \xrightarrow{\ f\ } \Pi_1$$
$$\searrow \quad \swarrow$$
$$\Pi$$

is a map in $(\mathscr{G}, \Pi)$ and M is a Π-module we get an exact sequence

$$\cdots \longrightarrow H_n(\Pi_0, M) \longrightarrow H_n(\Pi_1, M) \longrightarrow H_n(f, M) \longrightarrow H_{n-1}(\Pi_0, M) \longrightarrow \cdots$$

and a similar one in cohomology. The relative term arises either as in Section 2 or by viewing f as an object in the double comma category $(\Pi_0, \mathscr{G}, \Pi_1)$ and using this section. The equivalence results from the fact that the homology coproduct theorem holds for groups.

This sequence can be obtained topologically by considering the map of Eilenberg-Mac Lane spaces $K(\Pi_0, 1) \longrightarrow K(\Pi_1, 1)$. It is also obtained in [Takasu (1959/60)].

As a special case, if

$$\Pi \xrightarrow{\ f\ } \Pi/N$$
$$\searrow \quad \swarrow$$
$$1$$

is division by a normal subgroup and we take coefficients in $\mathbf{Z}$ as a 1-module, then $H_0(f) = 0$ and $H_1(f) \simeq N/[\Pi, N]$. Thus the Stallings-Stammbach sequence ([Stallings (1965), Stammbach (1966)]) falls out:

$$H_1(\Pi) \longrightarrow H_1(\Pi/N) \longrightarrow N/[\Pi, N] \longrightarrow H_0(\Pi) \longrightarrow H_0(\Pi/N) \longrightarrow 0$$

(our dimensional indices). Doubtless many of the other sequences of this type given in [Eckmann & Stammbach (1967)] can be got similarly.

8.3. Commutative rings and algebras. Given maps of commutative rings

$$A \longrightarrow B \longrightarrow C$$
$$\searrow \quad \downarrow \quad \swarrow$$
$$D$$

we obtain exact sequences

$$\cdots \longrightarrow H_n(A, M) \longrightarrow H_n(B, M) \longrightarrow H_n(A \longrightarrow B, M) \xrightarrow{\ \partial\ } H_{n-1}(A, M) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_n(A \longrightarrow B, M) \longrightarrow H_n(A \longrightarrow C, M) \longrightarrow H_n(B \longrightarrow C, M) \xrightarrow{\ \partial\ } H_{n-1}(A \longrightarrow B, M) \longrightarrow \cdots$$

for a D-module M; similar sequences are obtainable in cohomology. Taking $B = C = D$, and homology with respect to the cotriple $\mathbf{G}$ arising from

$$\mathbf{Comm} \xrightarrow{\ G\ } \mathbf{Comm}$$
$$\searrow \qquad \nearrow$$
$$\mathbf{Sets}$$

these sequences coincide with those of [Lichtenbaum & Schlessinger (1967), André (1967), Quillen (1967)], as a result of the following facts:

(a) $(A, \mathbf{G})$ is the cotriple arising from

$$(A, \mathbf{Comm}) \longrightarrow (A, \mathbf{Comm})$$
$$\underset{U}{\searrow} \qquad \underset{F}{\nearrow}$$
$$\mathbf{Sets}$$

where $(A \longrightarrow B)U = B$.

(b) If $E\colon(\mathbf{Comm}, D) \longrightarrow D\text{-}\mathbf{Mod}$ is $AE = \mathrm{Diff}_D(A) \otimes_D M$, then $(A \longrightarrow B)(A, E) = \Omega^1_{B/A} \otimes_B M$. If $E\colon(\mathbf{Comm}, D)^* \longrightarrow D\text{-}\mathbf{Mod}$ is $AE = \mathrm{Der}(A, M)_D$, then

$$(A \longrightarrow B)(A, E) = A\text{-}\mathrm{Der}(B, M)_D$$

(c)

$$H_n(A \longrightarrow B, E)_{\mathbf{G}} \overset{\sim}{\longrightarrow} H_n(A \longrightarrow B, (A, E))_{A,\mathbf{G}}$$

for any coproduct preserving coefficient functor $E\colon(\mathbf{Comm}, D) \longrightarrow \mathscr{A}$ (writing A for $A \longrightarrow D$).

(a) has been noted in Section 1. For (b),

$$\begin{aligned}
(A \longrightarrow B)(A, E) &= \mathrm{coker}(\mathrm{Diff}_D(A) \otimes_D M \longrightarrow \mathrm{Diff}_D(B) \otimes_D M) \\
&= \mathrm{coker}(\mathrm{Diff}_D(A) \longrightarrow \mathrm{Diff}_D(B)) \otimes_D M \\
&= (\Omega^1_{B/A} \otimes_B D) \otimes_D M \\
&= \Omega^1_{B/A} \otimes_B M.
\end{aligned}$$

In the dual theory, it is appropriate to lift a functor $E\colon\mathbf{C}^* \longrightarrow \mathscr{A}$ to a functor

$$(E, A)\colon(\mathbf{C}^*, A) \longrightarrow \mathscr{A}$$

by defining $(B \longrightarrow A)(E, A) = \ker(BE \longrightarrow AE)$. For E the contravariant functor in (b), we have then

$$\begin{aligned}
(A \longrightarrow B)(A, E) &= \ker(\mathrm{Der}(B, M)_D \longrightarrow \mathrm{Der}(A, M)) \\
&= A\text{-}\mathrm{Der}(B, M)_D.
\end{aligned}$$

Alternatively and of course equivalently, dualize the coefficient category of D-modules. Finally (c) follows from the fact that the coproduct theorem holds for homology in this category when one factor is free. Indeed, $C.(A, G)$ is the polynomial A-algebra $A[C]$ and is A-flat; thus the coproduct $(\ ) \otimes_A C.(A, G)$ preserves $(A, \mathbf{G})$-resolutions.

For the A-relative theory (1.2), the same exact sequences are available.

8.4. ASSOCIATIVE ALGEBRAS. Let $\mathbf{G}$ denote the cotriple on $K\text{-}\mathbf{Alg}$ arising out of the adjoint pair

$$K\text{-}\mathbf{Alg} \xrightarrow{\quad G \quad} K\text{-}\mathbf{Alg}$$
$$K\text{-}\mathbf{Mod}$$

If $\Lambda \in K\text{-}\mathbf{Alg}$, $\Gamma \longrightarrow \Lambda \in (K\text{-}\mathbf{Alg}, \Lambda)$ and M is a Λ-bimodule, we let $H_n(\Gamma, M)_{\mathbf{G}}$ and $H^n(\Gamma, M)_{\mathbf{G}}$ denote the derived functors with respect to $\mathbf{G}$ of $\mathrm{Diff}_\Lambda(\ ) \otimes_\Lambda eM$ and $\mathrm{Der}_\Lambda(\ , M)$ respectively. Let us drop Λ from the notation from now on. Hence $\Gamma \longrightarrow \Gamma_1$ below really refers to $\Gamma \longrightarrow \Gamma_1 \longrightarrow \Lambda$ etc.

THEOREM (8.5).

$$H_n(\Gamma \longrightarrow \Gamma_1, M)_{\mathbf{G}} \simeq H_n(\Gamma_1, M)_{(\Gamma, \mathbf{G})},$$
$$H^n(\Gamma \longrightarrow \Gamma_1, M)_{\mathbf{G}} \simeq H^n(\Gamma_1, M)_{(\Gamma, \mathbf{G})}.$$

Proof. According to (8.1) this requires showing that for any Γ',

$$H_n(\Gamma * \Gamma'G, M)_{\mathbf{G}} \simeq \begin{cases} H_0(\Gamma, M)_{\mathbf{G}} \oplus \mathrm{Der}(\Gamma'G, M), & n = 0 \\ H_n(\Gamma, M)_{\mathbf{G}}, & n > 0, \end{cases}$$

and similarly for cohomology. Before doing this we require

PROPOSITION. *Let* $\mathbf{G}_1$ *be the cotriple described in (6.3) above. Then*

$$H_n(\Gamma, M)_{\mathbf{G}} \simeq H_n(\Gamma, M)_{\mathbf{G}_1},$$
$$H^n(\Gamma, M)_{\mathbf{G}} \simeq H^n(\Gamma, M)_{\mathbf{G}_1}.$$

The proof will be given at the end of this section.

Now observe that any $\mathbf{G}$-projective is $\mathbf{G}_1$-projective and also is supplemented. Now $(\Gamma G_1^{n+1})_{n \geq 0}$ is a $\mathbf{G}_1$-resolution of Γ, which means it has a *unitary* K-linear contraction. As observed in (6.3) above, $(\Gamma G_1^{n+1} * \Gamma'G)_{n \geq 0}$ also has a unitary K-linear contraction and it clearly consists of $\mathbf{G}_1$-projectives. Thus it is a $\mathbf{G}_1$-resolution of $\Gamma * \Gamma'G$. But then

$$\mathrm{Diff}_\Lambda(\Gamma G_1^{n+1} * \Gamma'G)_{n \geq 0} \simeq \mathrm{Diff}_\Lambda(\Gamma G_1^{n+1})_{n \geq 0} \oplus \mathrm{Diff}_\Lambda \Gamma'G$$

the second summand being a constant simplicial object, and the result follows easily.

To prove (8.6) we use acyclic models in form (11.3) below with $\mathbf{G}_1$ as the comparison cotriple. First observe that there is a natural transformation $\varphi \colon G \longrightarrow G_1$ which actually induces a morphism of cotriples (meaning it commutes with both comultiplication and counit). Actually $\mathbf{G}_1$ is presented as a quotient of $\mathbf{G}$ and φ is the natural projection. Now we prove the theorem for cohomology. The proof for homology is similar. For any $\Gamma' \longrightarrow \Gamma$ and any Γ-bimodule M, let $\Gamma'E = \mathrm{Der}(\Gamma', M)$, $\Gamma'E^n = \mathrm{Der}(\Gamma'G_1^{n+1}, M)$. Let $\overline{\varphi} \colon \mathrm{Der}(\Gamma', M) \longrightarrow \Gamma'E$ be the identity and $\overline{\varphi}^n \colon \mathrm{Der}(\Gamma'G_1^{n+1}, M) \longrightarrow \Gamma'E^n$ be the map $\mathrm{Der}(\Gamma'\varphi^{n+1}, M)$. Define $\vartheta^n \colon \Gamma'G_1 E^n \longrightarrow \Gamma'E^n$ to be the composite

$$\mathrm{Der}(\Gamma'G_1 G^{n+1}, M) \xrightarrow{\mathrm{Der}(\Gamma'\varphi G^{n+1}, M)} \mathrm{Der}(\Gamma'G^{n+2}, M) \xrightarrow{\mathrm{Der}(\Gamma'\delta G^n, M)} \mathrm{Der}(\Gamma'G^{n+1}, M)$$

Then it is easily seen that $\mathrm{Der}(\Gamma'\varepsilon_1 G^{n+1}, M).\vartheta^n$ is the identity. (Of course, everything is dualised for cohomology.) Thus the proof is finished by showing that the complex

$$\cdots \longrightarrow \Gamma'G_1 E^n \longrightarrow \Gamma'G_1 E^{n-1} \longrightarrow \cdots \longrightarrow \Gamma'G_1 E^1 \longrightarrow \Gamma'G_1 E \longrightarrow 0$$

is exact. But the homology of that complex is simply the Hochschild homology of $\Gamma'G$, (with the usual degree shift), which in turn is $\mathrm{Ext}_{(\Gamma^e, K)}(\mathrm{Diff}_\Gamma \Gamma'G_1, M)$. Hence we complete

the proof by showing that $\mathrm{Diff}_\Gamma\Gamma'G$ is a K-relative Γ^e-projective. But $\mathrm{Der}(\Gamma'G_1, M)$ consists of those derivations of $\Gamma'G \longrightarrow M$ which vanish on the ideal of $\Gamma'G$ generated by $1_{\Gamma'} - 1_K$ or, since all derivations vanish on 1_K, it simply consists of those derivations which vanish on $1_{\Gamma'}$. But $\mathrm{Der}(\Gamma'G, M) \simeq \mathrm{Hom}_K(\Gamma', M)$ and it is easily seen that $\mathrm{Der}(\Gamma'G_1, M) \simeq \mathrm{Hom}_K(\Gamma'/K, M)$ where Γ'/K denotes $\mathrm{coker}(K \longrightarrow \Gamma)$. This in turn is $\simeq \mathrm{Hom}_\Lambda e(\Lambda^e \otimes \Gamma'/K, M)$ and so $\mathrm{Diff}_\Gamma\Gamma'G_1 \simeq \Lambda^e \otimes \Gamma'/K$ which is clearly a K-relative Λ^e-projective. This completes the proof.

9. Mayer-Vietoris theorems

Using assumptions about the homology of coproducts, we shall deduce some theorems of Mayer-Vietoris type. We learned of such theorems from André's work [André (1967)]. In the case of commutative algebras we obtain slightly more comprehensive results (9.5). Mostly, however, we concentrate on the case of groups (9.4).

Let $E: \mathbf{C} \longrightarrow \mathscr{A}$ be a coefficient functor.

THEOREM (9.1). *Let*

$$\begin{array}{ccc} X & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & Y \end{array}$$

be a pushout diagram in $\mathbf{C}$ and suppose that the homology coproduct theorem holds for Y viewed as a coproduct in $(X, \mathbf{C})$:

$$H_n(X \longrightarrow X_1, E) \oplus H_n(X \longrightarrow X_2, E) \overset{\sim}{\longrightarrow} H_n(X \longrightarrow Y, E), \quad n \geq 0.$$

Then there is an exact sequence,

$$\cdots \longrightarrow H_n(X, E) \longrightarrow H_n(X_1, E) \oplus H_n(X_2, E) \longrightarrow H_n(Y, E)$$

$$\overset{\partial}{}$$

$$H_{n-1}(X, E) \longrightarrow \cdots\cdots\cdots\cdots \longrightarrow H_0(Y, E) \longrightarrow 0$$

(The maps in the sequence are the usual Mayer-Vietoris maps $(\beta, -\gamma), (\beta_1, \gamma_1)$ transpose, if we momentarily write

$$\beta: H(X) \longrightarrow H(X_1), \qquad\qquad \beta_1: H(X_1) \longrightarrow H(Y),$$
$$\gamma: H(X) \longrightarrow H(X_2), \qquad\qquad \gamma_1: H(X_2) \longrightarrow H(Y).)$$

THEOREM (9.2). *Suppose that the natural map is an isomorphism*

$$H_n(X, E) \oplus H_n(Y, E) \xrightarrow{\sim} H_n(X * Y, E), \quad n \geq 0$$

for any map $X * Y \longrightarrow Z$ *there is an exact sequence*

$$\cdots \to H_n(Z, E) \to H_n(X \to Z, E) \oplus H_n(Y \to Z, E) \to H_n(X * Y \to Z, E)$$

$$H_{n-1}(Z, E) \longrightarrow \cdots\cdots\cdots \longrightarrow H_0(X * Y \to Z, E) \longrightarrow 0$$

(with ∂ labeling the diagonal arrow)

For the proof of (9.1), write down the diagram

$$
\begin{array}{ccccccc}
H(X, E) & \to & H(X_1, E) & \longrightarrow & H(X \to X_1, E) & \xrightarrow{\partial} & H(X, E) \\
= \downarrow & & \downarrow & & \downarrow & & \downarrow = \\
H(X, E) & \to & H(Y, E) & \to & H(X \to X_1, E) \oplus H(X \to X_2, E) & \xrightarrow{\partial} & H(X, E) \\
= \uparrow & & \uparrow & & \uparrow & & \uparrow = \\
H(X, E) & \to & H(X_2, E) & \longrightarrow & H(X \to X_2, E) & \xrightarrow{\partial} & H(X, E)
\end{array}
$$

All three triangles are exact, the middle one by the coproduct theorem in $(X, \mathbf{C})$. Lemma (9.3) below then yields that

$$H(X, E) \longrightarrow H(X_1, E) \oplus H(X_2, E) \longrightarrow H(Y, E) \longrightarrow H(X, E)$$

is an exact triangle. For (9.2), write

$$
\begin{array}{ccccccc}
H(Z, E) & \longrightarrow & H(X \to Z, E) & \xrightarrow{\partial} & H(X, E) & \longrightarrow & H(Z, E) \\
= \downarrow & & \downarrow & & \downarrow & & \downarrow = \\
H(Z, E) & \longrightarrow & H(X * Y \to Z, E) & \xrightarrow{\partial} & H(X, E) \oplus H(Y, E) & \longrightarrow & H(Z, E) \\
= \uparrow & & \uparrow & & \uparrow & & \uparrow = \\
H(Z, E) & \longrightarrow & H(Y \to Z, E) & \xrightarrow{\partial} & H(Y, E) & \longrightarrow & H(Z, E)
\end{array}
$$

and again apply

LEMMA (9.3). *In an abelian category*

is commutative with exact triangles for rows $\iff$

is commutative and has exact triangles for rows.

This lemma is dual to its converse and needn't be proved.

9.4 GROUPS. Theorem (9.2) holds without restriction. Because of the validity of the homology coproduct theorem (7.3), if $\Pi_0 * \Pi_1 \longrightarrow \Pi$ we get an exact sequence

$$\cdots \longrightarrow H_n(\Pi, M) \longrightarrow H_n(\Pi_0 \longrightarrow \Pi, M) \oplus H_n(\Pi_1 \longrightarrow \Pi, M)$$
$$\longrightarrow H_n(\Pi_0 * \Pi_1 \longrightarrow \Pi, M) \longrightarrow H_{n-1}(\Pi, M) \longrightarrow \cdots$$

if M is a Π-module; similar sequences hold in cohomology, or in homology with coefficients in any coproduct-preserving functor.

As to (9.1), its applicability is a little more restricted. Suppose that Π_0 is a subgroup of Π_1 and Π_2 and that Π is the pushout or amalgamated coproduct:

It will be shown that if M is a Π-module, then the Π_0-coproduct theorem holds for homology:

$$H_n(\Pi_0 \longrightarrow \Pi_1, M) \oplus H_n(\Pi_0 \longrightarrow \Pi_2, M) \overset{\sim}{\longrightarrow} H_n(\Pi_0 \longrightarrow \Pi, M).$$

Then the Mayer-Vietoris sequence

$$\cdots \longrightarrow H_n(\Pi_0, M) \longrightarrow H_n(\Pi_1, M) \oplus H_n(\Pi_2, M) \longrightarrow H_n(\Pi, M)$$

$$\longrightarrow H_{n-1}(\Pi_0, M) \longrightarrow H_{n-1}(\Pi_1, M) \oplus H_{n-1}(\Pi_2, M) \longrightarrow H_{n-1}(\Pi, M) \longrightarrow \cdots$$

is exact. There is a similar exact sequence for cohomology with coefficients in M. While our argument will involve Diff_Π, we cannot claim this for arbitrary coefficient functors but only for those that are a composition of $\mathrm{Diff}_\Pi \colon (\mathscr{G}, \Pi) \longrightarrow \Pi\text{-}\mathbf{Mod}$ and an additive functor $E \colon \Pi\text{-}\mathbf{Mod} \longrightarrow \mathscr{A}$. This theorem also has a topological proof using Eilenberg-Mac Lane spaces. Similar results have been obtained by [Ribes (1967)]. We now launch into the algebraic details:

The free group cotriple preserves monomorphisms: let Y_n be the pushout or amalgamated coproduct

$$
\begin{array}{ccc}
\Pi_0 G^{n+1} & \longrightarrow & \Pi_1 G^{n+1} \\
\downarrow & & \downarrow \\
\Pi_2 G^{n+1} & \longrightarrow & Y_n
\end{array}
\qquad (n \geq -1)
$$

Thus (Y_n) is an augmented simplicial group, with $Y_{-1} = \Pi$. Moreover Y_n is a free group when $n \geq 0$ as $Y_n = SF$ where S is the set-theoretic pushout

$$
\begin{array}{ccc}
\Pi_0 G^n U & \longrightarrow & \Pi_1 G^n U \\
\downarrow & & \downarrow \\
\Pi_2 G^n U & \longrightarrow & S
\end{array}
$$

and F is the free group functor $\mathscr{S} \longrightarrow \mathscr{G}$, which as a left adjoint preserves pushouts. Applying Diff_Π, we get a square

(a)
$$
\begin{array}{ccc}
\mathrm{Diff}_\Pi(\Pi_0 G^{n+1}) & \longrightarrow & \mathrm{Diff}_\Pi(\Pi_1 G^{n+1}) \\
\downarrow & & \downarrow \\
\mathrm{Diff}_\Pi(\Pi_2 G^{n+1}) & \longrightarrow & \mathrm{Diff}_\Pi(Y_n)
\end{array}
$$

which is exact, i.e., simultaneously a pushout and a pullback. We will prove this later, as also fact (b) arrayed below. For the rest of this section we write

$$\mathrm{Diff} = \mathrm{Diff}_\Pi.$$

Now using the usual Mayer-Vietoris maps we get an exact sequence of chain complexes

$$0 \longrightarrow (\mathrm{Diff}(\Pi_0 G^{n+1})) \longrightarrow (\mathrm{Diff}(\Pi_1 G^{n+1}) \oplus \mathrm{Diff}(\Pi_2 G^{n+1})) \longrightarrow (\mathrm{Diff}(Y_n)) \longrightarrow 0$$

for $n \geq 0$, whence the homology sequence

(b)

$$
\begin{array}{ccccc}
\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow H_p(\mathrm{Diff}(Y_*)) \\
0 \longleftarrow \cdots \longrightarrow H_1(\mathrm{Diff}(Y_*)) \\
\mathrm{Diff}(\Pi_0) \longrightarrow \mathrm{Diff}(\Pi_1) \oplus \mathrm{Diff}(\Pi_2) \longrightarrow H_0(\mathrm{Diff}(Y_*)) \longrightarrow 0
\end{array}
$$

(The p illustrated is ≥ 2); in addition, the map ∂ is zero. This yields the conclusion that $(\mathrm{Diff}(Y_n))$, $n \geq -1$, is a Π-free resolution of $\mathrm{Diff}(\Pi) = I\Pi$ in the category of Π-modules.

Let E be any additive functor $\Pi\text{-}\mathbf{Mod} \longrightarrow \mathscr{A}$. The first two columns of the following commutative diagram are exact, hence the third column which consists of the mapping cones of the horizontal maps is also exact.

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Diff}(\Pi_0 G^{n+1})E & \longrightarrow & \mathrm{Diff}(\Pi_0 G^{n+1})E & \longrightarrow & C' \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Diff}(\Pi_0 G^{n+1})E \oplus \mathrm{Diff}(\Pi_0 G^{n+1})E & \twoheadrightarrow & \mathrm{Diff}(\Pi_1 G^{n+1})E \oplus \mathrm{Diff}(\Pi_2 G^{n+1})E & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Diff}(\Pi_0 G^{n+1})E & \longrightarrow & \mathrm{Diff}(Y_*)E & \longrightarrow & C'' \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

for $n \geq 0$. C' is acyclic as it is the mapping cone of an identity. Clearly,

$$H_n(C) = H_n(\Pi_0 \longrightarrow \Pi_1, E) \oplus H_n(\Pi_0 \longrightarrow \Pi_2, E) \quad \text{and}$$
$$H_n(C'') = H_n(\Pi_0 \longrightarrow \Pi, E).$$

The homology sequence of $0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$ then proves the coproduct theorem for groups under Π_0. This completes the proof that (9.1) applies to amalgamated coproduct diagrams in $\mathscr{G}$, modulo going back and proving (a), (b).

Square (a) is obviously a pushout since $\mathrm{Diff}_\Pi \colon (\mathscr{G}, \Pi) \longrightarrow \Pi\text{-}\mathbf{Mod}$ is a left adjoint and preserves pushouts. The hard part is proving that it is a pullback. For that it is enough to show that the top map $\mathrm{Diff}_\Pi(\Pi_0 G^{n+1}) \longrightarrow \mathrm{Diff}_\Pi(\Pi_1 G^{n+1})$ is a monomorphism, in view of:

LEMMA. *In an abelian category,*

$$\begin{array}{ccc} A_0 & \xrightarrow{\alpha_1} & A_1 \\ \alpha_2 \downarrow & & \downarrow \beta_1 \\ A_2 & \xrightarrow{\beta_2} & A_3 \end{array}$$

is a pushout $\Longleftrightarrow$

$$A_0 \xrightarrow{(\alpha_1, \alpha_2)} A_1 \oplus A_2 \xrightarrow{\left(\begin{smallmatrix} \beta_1 \\ -\beta_2 \end{smallmatrix}\right)} A_3 \longrightarrow 0$$

is exact, and dually, is a pullback $\Longleftrightarrow$

$$0 \longrightarrow A_0 \xrightarrow{(\alpha_1, \alpha_2)} A_1 \oplus A_2 \xrightarrow{\left(\begin{smallmatrix} \beta_1 \\ -\beta_2 \end{smallmatrix}\right)} A_3$$

is exact.

This is standard. Thus, we are reduced to:

LEMMA. *If* $\Pi_0 \longrightarrow \Pi$ *is a subgroup, then* $\mathrm{Diff}_\Pi(\Pi_0) \longrightarrow \mathrm{Diff}_\Pi(\Pi)$ *is a monomorphism of* Π*-modules. If*

$$\begin{array}{ccc} \Pi_0 & \longrightarrow & \Pi_1 \\ & \searrow \quad \swarrow & \\ & \Pi & \end{array}$$

is a diagram of subgroups, then

$$\begin{array}{ccc} \mathrm{Diff}_\Pi(\Pi_0) & \longrightarrow & \mathrm{Diff}_\Pi(\Pi_1) \\ & \searrow \quad \swarrow & \\ & \mathrm{Diff}_\Pi(\Pi) & \end{array}$$

commutes, hence $\mathrm{Diff}_\Pi(\Pi_0) \longrightarrow \mathrm{Diff}_\Pi(\Pi_1)$ *is a monomorphism of* Π*-modules.*

PROOF. [1] We write $x \in \Pi_0$, $y \in \Pi$ and present an isomorphism

$$\begin{array}{ccc} \mathrm{Diff}_\Pi(\Pi_0) = \mathbb{Z}\Pi \otimes_{\Pi_0} I\Pi_0 & \xrightarrow{\quad f \quad} & D \\ & \searrow \quad \swarrow & \\ & I\Pi & \end{array}$$

---

[1] There is a simple exact-sequences argument.

where D is the Π-submodule generated by all $x - 1$. f and f^{-1} are the Π-linear maps determined by the correspondence $1 \otimes (x - 1) \Leftrightarrow x - 1$. f is more-or-less obviously well-defined. As for f^{-1}, it is deduced from the exact sequence of Π-modules

$$R \xrightarrow{\partial_1} F \xrightarrow{\partial_0} D \longrightarrow 0$$

$$f_0 \downarrow \quad \swarrow f^{-1}$$

$$\mathbb{Z}\Pi \otimes_{\Pi_0} I\Pi_0$$

where F is the free Π-module on generators $[x]$, $[x]\partial_0 = x - 1$, and ∂_1 is the sub-module generated by all elements of the form

$$y[x] + y_1[x_1] - y_1[y_1^{-1}yx]$$

where $y = y_1 x_1$. f_0 is defined by $[x]f_0 = 1 \otimes (x - 1)$ and annihilates ∂_1.

For the proof of the statements around (b), we know that

$$H_p(\mathrm{Diff}_{\Pi_0}(\Pi_0 G^{n+1})_{n \geq 0}) = \begin{cases} \mathrm{Diff}_{\Pi_0}(\Pi_0) & p = 0 \\ 0 & p > 0 \end{cases}$$

by (1.2). After tensoring over Π_0 with $\mathbb{Z}\Pi$, which is Π_0-projective since $\Pi_0 \longrightarrow \Pi$ is a sub-group, we find that the homology becomes $\mathrm{Diff}_\Pi(\Pi_0)$ in dimension 0 and 0 in dimensions > 0. This accounts for the two columns of 0's in (b). The fact that $\partial = 0$ results from exactness and the above Lemma, which implies that $\mathrm{Diff}_\Pi(\Pi_0) \longrightarrow \mathrm{Diff}_\Pi(\Pi_1) \oplus \mathrm{Diff}_\Pi(\Pi_2)$ is monomorphic. This completes the proof.

(9.5) COMMUTATIVE ALGEBRAS. If

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & B \otimes_A C \end{array}$$

is a pushout in the category of commutative K-algebras, where K is a commutative ring and M is a $B \otimes_A C$-module, then

$$H_n(A \longrightarrow B, M) \oplus H_n(A \longrightarrow C, M) \xrightarrow{\sim} H_n(A \longrightarrow B \otimes_A C, M)$$

for $n \geq 0$ if $\mathrm{Tor}_p^A(B, C) = 0$ for $p > 0$ (homology with respect to the absolute cotriple in the category of commutative K-algebras (cf. (7.4)). In this case (9.1) gives an exact sequence

$$\cdots \longrightarrow H_n(A, M) \longrightarrow H_n(B, M) \oplus H_n(C, M) \longrightarrow H_n(B \otimes_A C, M) \longrightarrow H_{n-1}(A, M) \longrightarrow \cdots$$

A similar sequence holds in cohomology under the same Tor assumption. If $K = A$, this coincides with the homology coproduct theorem.

If $A \otimes_K B \longrightarrow C$ is a K-algebra map and $\mathrm{Tor}_p^K(A, B) = 0$ for $p > 0$, then the homology assumption in (9.2) is satisfied and we get the sequence

$$\cdots \longrightarrow H_n(C, M) \longrightarrow H_n(A \longrightarrow C, M) \oplus H_n(B \longrightarrow C, M)$$
$$\longrightarrow H_n(A \otimes_K B \longrightarrow C, M) \longrightarrow H_{n-1}(C, M) \longrightarrow \cdots$$

if M is a C-module; similarly in cohomology. This is the same sequence as in [André (1967)], Section 5, but the assumption $\mathrm{Tor}_p^K(C, C) = 0$, $p > 0$ employed there is seen to be superfluous.

10. Cotriples and models

For our purposes it is sufficient to consider a **category with models** to be a functor $\mathbf{M} \longrightarrow \mathbf{C}$ where $\mathbf{M}$ is discrete. The objects of $\mathbf{M}$ are known as the **models**. Many cotriples can be constructed in the following manner.

(10.1) MODEL-INDUCED COTRIPLE. If $X \in \mathbf{C}$ let

$$XG = \bigstar_{\substack{M \longrightarrow X \\ M \in \mathbf{M}}} M$$

the coproduct indexed by all maps of model objects $M \longrightarrow X$.

We assume that such coproducts exist in $\mathbf{C}$, and write $M \longrightarrow X$ instead of $MI \longrightarrow X$ in order to avoid having to name $I \colon \mathbf{M} \longrightarrow \mathbf{C}$

Let $\langle x \rangle \colon M \longrightarrow XG$ denote the canonical map of the cofactor indexed by a map $x \colon M \longrightarrow X$. Then

$$XG \xrightarrow{\ X\varepsilon\ } X$$

is the map such that $\langle x \rangle X\varepsilon = x$ for all $x \colon M \longrightarrow X$, $M \in \mathbf{M}$.

$$XG \xrightarrow{\ X\delta\ } XGG$$

is the map such that $\langle x \rangle X\delta = \langle\langle x \rangle\rangle$ for all such x. (Since $\langle x \rangle \colon M \longrightarrow XG$, $\langle\langle x \rangle\rangle \colon M \longrightarrow (XG)G$.) Both ε and δ are natural transformations, and as

$$\langle x \rangle X\delta.XG\delta = \langle\langle\langle x \rangle\rangle\rangle = \langle x \rangle X\delta.X\delta G \qquad \text{and}$$
$$\langle x \rangle X\delta.XG\varepsilon = \langle\langle x \rangle\rangle XG\varepsilon = \langle x \rangle = \langle\langle x \rangle\rangle X\varepsilon G = \langle x \rangle X\delta.X\varepsilon G$$

we have that $\mathbf{G} = (\mathbf{G}, \varepsilon, \delta)$ is a cotriple in $\mathbf{C}$, which we call **model-induced**. (This special case is dual to the "triple structure" which Linton discusses in [Linton (1969)]; see also [Appelgate & Tierney (1969)].)

If M is a model, then M viewed as an object in $\mathbf{C}$ is **G**-projective (even a **G**-coalgebra):

$$M \xrightarrow{\ \langle M \rangle\ } MG \xrightarrow{\ M\varepsilon\ } M.$$

Some other relations between model concepts and cotriple concepts are: A simplicial object X_*, has zero homotopy relative to $\mathbf{G}$ (every (AG, X_*) has zero homotopy) $\Longleftrightarrow$ every simplicial set (M, X_*) has zero homotopy. In the additive case, $\mathbf{G}$-acyclicity is equivalent to acyclicity relative to all of the objects $M \in \mathbf{M}$.

(10.2) EXAMPLES OF MODEL-INDUCED COTRIPLES. (a) Let $1 \longrightarrow R\text{-}\mathbf{Mod}$ be the functor whose value is R. Then $AG = \oplus R$, over all elements $R \longrightarrow A$, is the free R-module cotriple (1.1). More generally, if $\mathbf{C}$ is tripleable over sets and $1 \longrightarrow \mathbf{C}$ has value $1F$, the free object on 1 generator, then the model-induced cotriple $\mathbf{G}$ is the free cotriple in $\mathbf{C}$, e.g., $\mathbf{C} = K\text{-}\mathbf{Alg}$, $1F = K[x]$, $\mathbf{C} = \mathbf{Groups}$, $1F = \mathbb{Z}$.

(b) Let $1 \longrightarrow \mathbf{Ab}$ have value $\mathbb{Q}/\mathbb{Z}$ (rationals mod one). Let T be the model-induced **triple** in $\mathbf{Ab}$

$$AT = \prod_{A \longrightarrow \mathbb{Q}/\mathbb{Z}} \mathbb{Q}/\mathbb{Z}.$$

$(AT^{n+1})_{n \geq -1}$ is an injective resolution of A. The composition

$$R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab} \xrightarrow{\;T\;} \mathbf{Ab} \xrightarrow{\;\mathrm{Hom}_{\mathbb{Z}}(R,\text{-})\;} R\text{-}\mathbf{Mod}$$

is the **Eckmann-Schopf** triple T_R in $R\text{-}\mathbf{Mod}$. $(AT^{n+1})_{n \geq -1}$ is an R-injective resolution of an R-module A.

(c) Let $\mathbf{M} \longrightarrow R\text{-}\mathbf{Mod}$ be the subset of cyclic R-modules. The model-induced cotriple is the **pure** cotriple

$$CG = \bigoplus_{\substack{R/I \longrightarrow C \\ I \subseteq R \\ I \neq 0}} R/I.$$

The $\mathbf{G}$-homology and cohomology of $C \in R\text{-}\mathbf{Mod}$ with coefficients in $A \otimes_R (\text{-})$, resp. $\mathrm{Hom}_R(\text{-}, A)$, are Harrison's $\mathrm{Ptor}_n^R(A, C)$, $\mathrm{Pext}_R^n(C, A)$; Pext classifies pure extensions of R-modules [Harrison (1959)]. This example is one of the original motivations for relative homological algebra.

(d) Let $\boldsymbol{\Delta} \longrightarrow \mathbf{Top}$ be the discrete subcategory whose objects are the standard Euclidian simplices Δ_p, $p \geq 0$. Then

$$XG = \bigcup_{\substack{\Delta_p \longrightarrow X \\ p \geq 0}} \Delta_p$$

If $\mathbf{Top} \xrightarrow{\;E\;} \mathscr{A}$ is $H_0(\text{-}, M)_{\mathrm{sing}}$, the 0-th singular homology group of X with coefficients in M, then

$$H_n(X, H_0(\text{-}, M)_{\mathrm{sing}})_{\mathbf{G}} \simeq H_n(X, M)_{\mathrm{sing}}.$$

This is proved by a simple acyclic-models argument (11.2) or equivalently by collapsing of a spectral sequence like that in (10.5) . Singular cohomology is similarly captured.

(e) Let $\boldsymbol{\Delta} \longrightarrow \mathrm{Simp}$ be the discrete subcategory of all Δ_p, $p \geq 0$, where Simp is the category of simplicial spaces. The model-induced cotriple is $XG = \cup \Delta_p$ over all simplicial maps $\Delta_p \longrightarrow X$, $p \geq 0$. The $\mathbf{G}$-homology is simplicial homology.

(10.3) HOMOLOGY OF A CATEGORY. In [Roos (1961)], [André (1965)] Roos and André defined a homology theory $H_n(\mathbf{X}, E)$ of a category $\mathbf{X}$ with coefficients in a functor $E: \mathbf{X} \longrightarrow \mathscr{A}$. The homology theory arises from a complex

$$C_n(\mathbf{X}, E) = \sum_{M_0 \xrightarrow{\alpha_0} M_1 \longrightarrow \cdots \xrightarrow{\alpha_{n-1}} M_n} M_0 E, \qquad n \geq 0.$$

Using the $\langle\rangle$ notation for the coproduct injections $M_0 E \longrightarrow C_n(\mathbf{X}, E)$, the face operators $\varepsilon_i: C_n \longrightarrow C_{n-1}$, $0 \leq i \leq n$, are

$$\langle \alpha_0, \ldots, \alpha_{n-1} \rangle \varepsilon_i = \begin{cases} \alpha_0 E . \langle \alpha_1, \ldots, \alpha_{n-1} \rangle, & i = 0 \\ \langle \alpha_0, \ldots, \alpha_{i-1}\alpha_i, \ldots, \alpha_{n-1} \rangle, & 0 < i < n \\ \langle \alpha_0, \ldots, \alpha_{n-2} \rangle, & i = n; \end{cases}$$

it is understood that $\langle \alpha_0 \rangle \varepsilon_0 = \alpha_0 E . \langle M_1 \rangle$, $\langle \alpha_0 \rangle \varepsilon_1 = \langle M_0 \rangle$ and $C_0(\mathbf{X}, E) = \sum ME$ over all $M \in \mathbf{X}$. The homology groups of this complex, with respect to the boundary operator $\partial = \sum (-1)^i \varepsilon_i$, are denoted by $H_n(\mathbf{X}, E)$.

Clearly, $H_0(\mathbf{X}, E) = \varinjlim E$, and Roos proves that if $\mathscr{A}$ has exact direct sums (AB4), then $H_n(\mathbf{X}, E) = (L_n \varinjlim)(E)$, the left satellite of the direct limit functor $(\mathbf{X}, \mathscr{A}) \longrightarrow \mathscr{A}$, for $n > 0$.

If there is a terminal object $1 \in \mathbf{X}$, then $H_n(\mathbf{X}, E) = 0$ for $n > 0$. This follows from the existence of homotopy operators

$$C_0 \xrightarrow{h_0} C_1 \longrightarrow \cdots \longrightarrow C_n \xrightarrow{h_n} \cdots$$

defined by $\langle \alpha_0, \ldots, \alpha_n \rangle h_n = \langle \alpha_0, \ldots, \alpha_n, (\text{-}) \rangle$ where (-) is the unique map of the appropriate object into 1. This is also obvious from the fact that $\varinjlim E = 1E$, that $\varinjlim$ is an exact functor (assuming $\mathscr{A}$ is AB4).

More generally, if $\mathbf{X}$ is directed and $\mathscr{A}$ is AB5, then $H_n(\mathbf{X}, E) = 0$ for $n > 0$. "Directed" means that if $X_0, X_1 \in \mathbf{X}$, then there exist an object $X \in \mathbf{X}$, maps $X_0 \longrightarrow X \longleftarrow X_1$, and if $x, y: X_1 \longrightarrow X_0$, then there exists a map $z: X_0 \longrightarrow X$, such that $xz = yz$. AB5 is equivalent to exactness of direct limits over directed index categories.

(10.4) ANDRÉ-APPELGATE HOMOLOGY. In a models situation, let $\operatorname{Im} \mathbf{M}$ be the full subcategory of $\mathbf{C}$ generated by the image of $\mathbf{M} \longrightarrow \mathbf{C}$. If $X \in \mathbf{C}$, $(\operatorname{Im} \mathbf{M}, X)$ is the category whose objects are maps of models $M \longrightarrow X$ and whose maps are triangles $X \longleftarrow M_0 \longrightarrow M_1 \longrightarrow X$. If $E_0: \operatorname{Im} \mathbf{M} \longrightarrow \mathscr{A}$ is a coefficient functor, E_0 can be construed as a functor $(\operatorname{Im} \mathbf{M}, X) \longrightarrow \mathscr{A}$ by $(M \longrightarrow X)E_0 = ME_0$.

The **André-Appelgate homology** of X *with coefficients in* E_0 (relative to the models $\mathbf{M} \longrightarrow \mathbf{C}$) is

$$A_n(X, E_0) = H_n[(\operatorname{Im} \mathbf{M}, X), E_0]$$

where on the right we have the Roos-André homology of the comma category. Explicitly, the chain complex which gives rise to this homology theory has

$$C_n(\mathbf{X}, E_0) = \sum_{M_0 \xrightarrow{\alpha_0} M_1 \longrightarrow \cdots \longrightarrow M_n \xrightarrow{x} X} M_0 E$$

with boundary operator as in [André (1967)], Section 1. We note that H. Appelgate [Appelgate (1965)] developed this homology theory in a different way. He viewed the above complex as being generated by its 0-chains acting as a cotriple in the functor category $(\mathbf{C}, \mathscr{A})$.

A basic property of this theory is that if M is a model, then

$$A_n(M, E_0) \simeq \begin{cases} ME_0, & n = 0 \\ 0, & n > 0, \end{cases}$$

for any functor $E_0\colon \operatorname{Im}\mathbf{M} \longrightarrow \mathscr{A}$. The category $(\operatorname{Im}\mathbf{M}, M)$ has M as final object and the contracting homotopy in (10.3) in available [André (1967)], Sub-section 1.1.

In general,

$$A_0(X, E_0) = X.E_J(E_0)$$

where $E_J\colon (\operatorname{Im}\mathbf{M}, \mathscr{A}) \longrightarrow (\mathbf{C}, \mathscr{A})$, the **Kan extension**, is left adjoint to the restriction functor $(\operatorname{Im}\mathbf{M}, \mathscr{A}) \longleftarrow (\mathbf{C}, \mathscr{A})$. As the Kan extension can also be written as $\varinjlim(E_0\colon (\operatorname{Im}\mathbf{M}, X) \longrightarrow \mathscr{A})$, Roos's result implies that

$$A_n(X, E_0) = X.(L_n E_J)(E_0), \qquad n > 0.$$

provided that $\mathscr{A}$ is AB4. (For further information about Kan extension, see Ulmer's paper in this volume.)

The theory $A_n(X, E)$ is also defined when $E\colon \mathbf{C} \longrightarrow \mathscr{A}$ by restricting E to $\operatorname{Im}\mathbf{M}$. It can always be assumed that the coefficient functor is defined on all of $\mathbf{C}$. If not, take the Kan extension. The restriction of $E_J(E_0)$ to $\operatorname{Im}\mathbf{M}$ is equivalent to the given E_0 since $J\colon \operatorname{Im}\mathbf{M} \longrightarrow \mathbf{C}$ is full.

Now suppose we have both a models situation $\mathbf{M} \longrightarrow \mathbf{C}$ and a cotriple $\mathbf{G}$ in $\mathbf{C}$. To compare the homology theories $A_n(X, E)$ and $H_n(X, E) = H_n(X, E)_\mathbf{G}$, we use:

(10.5) SPECTRAL SEQUENCE. Suppose that all models $M \in \mathbf{M}$ are $\mathbf{G}$-projective. Then there is a spectral sequence

$$H_p(X, A_q(\text{-}, E)) \Longrightarrow A_{p+q}(X, E)$$

where the total homology is filtered by levels $\leq p$.

PROOF. For each $M \in \mathbf{M}$ choose a map $M\sigma\colon M \longrightarrow MG$ such that $M\sigma.M\varepsilon = M$. Define $\vartheta_q\colon C_q(X, E) \longrightarrow C_q(XG, E)$ by the identity map from the $\langle \alpha_0, \ldots, \alpha_{q-1}, x\rangle$-th summand to the $\langle \alpha_0, \ldots, \alpha_{q-1}.M_q\sigma, xG\rangle$-th. This makes $C_*(X, E)$ $\mathbf{G}$-representable, and the result follows from (11.3).   ∎

PROPOSITION (10.6). *If each category* $(\operatorname{Im}\mathbf{M}, XG)$ *is directed and the coefficient category* $\mathscr{A}$ *is AB5, then the above spectral sequence collapses and gives edge isomorphisms*

$$H_n(X, A_0(\text{-}, E)) \xrightarrow{\ \sim\ } A_n(X, E), \qquad n > 0.$$

The André-Appelgate theory has a natural augmentation $A_0(\text{-}, E) \longrightarrow E$, which is induced by the following cokernel diagram and map e such that $\langle x \rangle e = xE$:

$$C_1(X, E) \longrightarrow C_0(X, E) \longrightarrow A_0(X, E) \longrightarrow 0$$

with map e from $C_0(X,E)$ down to XE.

We obtain isomorphisms $H_n \simeq A_n$ from (10.6) when the augmentation is an isomorphism.

PROPOSITION (10.7). *Equivalent are:*

(1) $A_0(\text{-}, E) \longrightarrow E$ *is an isomorphism,*

(2) $E = E_J(E_0)$,   *where* $E_0 \colon \operatorname{Im}\mathbf{M} \longrightarrow \mathscr{A}$

(3) $E = E_J(E)$,   *the Kan extension of E restricted to* $\operatorname{Im}\mathbf{M}$.

Finally, (1) (2) (3) are implied by:

(4) *E commutes with direct limits and* $\operatorname{Im}\mathbf{M} \longrightarrow \mathbf{C}$ *is adequate [Isbell (1964)] /dense [Ulmer (1968)].*

The equivalences are trivial in view of fullness of $J \colon \operatorname{Im}\mathbf{M} \longrightarrow \mathbf{C}$. As to (4), this results from the fact that J is adequate/dense $\iff \varinjlim[(\operatorname{Im}\mathbf{M}, X) \longrightarrow \mathbf{C}] = X$ for all $X \in \mathbf{C}$

(10.8) EXAMPLES IN WHICH THE MODELS ARE G-PROJECTIVE. (a) Let the models be the values of the cotriple, that is, all $XG, X \in \mathbf{C}$. The comma category $(\operatorname{Im}\mathbf{M}, XG)$ has XG as terminal object, hence is directed. Thus $A_q(XG, E) = 0$ for $q > 0$ and any E, and (10.6) gives an isomorphism

$$H_n(XA_0(\text{-}, E)) \xrightarrow{\ \sim\ } A_n(X, E), \qquad n \geq 0.$$

(10.7) is inapplicable in general.

A stronger result follows directly from acyclic models (11.2). The complex $C_*(X, E)$ is $\mathbf{G}$-representable (10.5) and is $\mathbf{G}$-acyclic since each XG is a model. Thus

$$H_n(X, E) \xrightarrow{\ \sim\ } A_n(X, E)$$

(b) Another convenient set of models with the same properties is that of all $\mathbf{G}$-projectives.

Here and above, existence of the André-Appelgate complex raises some difficulties. The sets of models are too large. However, for coefficient functors with values in AB5

categories with generators, the problem can be avoided. Such categories are **Ab**-topos [Roos (1961)], realizable as categories of abelian sheaves on suitable *sites*, and it suffices to pass to models or abelian groups in a larger universe. (See the discussion of this point in [André (1967)] as well.)

(c) Let **G** be the free R-module cotriple and let $\mathbf{M} \longrightarrow R\text{-}\mathbf{Mod}$ be the set of finitely generated free R-modules. The categories $(\operatorname{Im}\mathbf{M}, XG)$ are directed, since any $M \longrightarrow XG$ can be factored

$$
\begin{array}{ccc}
M & \longrightarrow & XG = XUF \\
& \searrow \quad \swarrow {\scriptstyle iF} & \\
& SF &
\end{array}
$$

where $i\colon S \longrightarrow XU$ is a finite subset of the free basis XU, (U is the underlying set functor.) Moreover, $\operatorname{Im}\mathbf{M} \longrightarrow R\text{-}\mathbf{Mod}$ is adequate. Thus if E is any cocontinuous coefficient functor with values in an AB5 category, then $H_n(X, E) \simeq A_n(X, E)$ for all R-modules X.

(d) More generally, if **C** is tripleable over sets, $\aleph$ is a rank of the triple [Linton (1966a)] and **M** is the set of free algebras on fewer than $\aleph$ generators, then $(\operatorname{Im}\mathbf{M}, XG)$ can be proved directed in the same way, Thus homology relative to the models agrees with the cotriple homology (for cocontinuous AB5–category-valued coefficient functors; **G** is the free algebra cotriple relative to sets).

In these examples, adequacy/denseness of $\operatorname{Im}\mathbf{M}$ is well known or easily verified. In the following case adequacy fails. Let $1 \longrightarrow R\text{-}\mathbf{Mod}$ have R as value. $A_0(X, id.) \longrightarrow X$ is non-isomorphic (coefficients are in the identity functor $R\text{-}\mathbf{Mod} \longrightarrow R\text{-}\mathbf{Mod}$). In fact $A_0(X, id.) = R(X)/I$, the free R-module on X modulo the submodule generated by all $r(x) - (rx)$. Of course, $H_0(X, id.) \simeq X$ (homology with respect to the absolute cotriple, which is induced by the above model).

(e) COHOMOLOGY. Let $E\colon \mathbf{C} \longrightarrow \mathscr{A}^*$ be a "contravariant" coefficient functor. Isomorphisms $A^n(X, E) \overset{\sim}{\longrightarrow} H^n(X, E)$ follow purely formally in cases (a), (b) above. Cases (c), (d) offer the difficulty that the coefficient category $\mathscr{A}^*$ cannot be assumed to be AB5, since in practice it is usually dual to a category of modules and therefore AB 5^*. Assume that the rank $\aleph$ of the triple **T** is $\aleph_0$, however, one can proceed as follows.

If X is a **T**-algebra, the category of X-modules is abelian, AB5, has a projective generator and is complete and cocomplete. Thus injective resolutions can be constructed, in the abelian category sense. Moreover, the free abelian group functor $\operatorname{Diff}_X\colon (\mathbf{C}, X) \longrightarrow X\text{-}\mathbf{Mod}$ exists. Consider the André-Appelgate complex with values in $X\text{-}\mathbf{Mod}\colon (C_p(X)) = (C_p(X, \operatorname{Diff}_X)_{p \geq 0})$. Its homology, written $A_p(X)$, measures the failure of the André-Appelgate theory to be a derived functor on the category $X\text{-}\mathbf{Mod}$. If $Y \longrightarrow X$ is an X-module, $(Hom_X(C_p X, Y)_{p \geq 0})$ has $A^P(X, Y)$ as its cohomology. Let $(Y^q)_{q \geq 0}$ be an injective resolution of Y. We get a double complex $(Hom_X(C_p X, Y^q)_{p,q \geq 0})$, hence a universal-coefficients spectral sequence

$$
\operatorname{Ext}_X^q(A_p(X), Y) \Longrightarrow A^{p+q}(X, Y),
$$

where the total cohomology is filtered by q. (Use the fact that the complex $C_p(X)$ consists of projective X-modules.)

For example, in the case of commutative A-algebras over B, one obtains

$$\mathrm{Ext}_B^q(H_p(A,B,B),M) \longrightarrow H^{p+q}(A,B,M)$$

in the notation of [André (1967)], Section 16.

Similarly, in the cotriple theory, there is a spectral sequence

$$\mathrm{Ext}_X^q(H_p(X),Y) \longrightarrow H^{p+q}(X,Y)$$

Now, by the assumption that the rank of the triple is $\aleph_0$, the free **T**-algebra $XG \longrightarrow X$ is a filtered direct limit of free **T**-algebras of finite type, that is, of models. Since the homology $A_p(\_)$ commutes with filtered limits, $A_p(XG) = 0$ for $p > 0$, $A_0(XG) = \mathrm{Diff}_X(XG)$. Thus the above spectral sequence yields $A^n(XG,Y) = 0$ for $n > o$, $A^0(XG,Y) = \mathrm{Hom}_X(XG,Y)$. Acyclic models (11.2) now yields isomorphisms

$$A^n(X,Y) \overset{\sim}{\longrightarrow} H^n(X,Y).$$

A case in which this comparison technique runs into difficulty is the following. Let $\mathbf{M} \longrightarrow K\text{-}\mathbf{Alg}$ be the set of tensor algebras of finitely generated K-modules, and let **G** be the cotriple in $K\text{-}\mathbf{Alg}$ relative to K-modules. Homology isomorphisms $H_n \overset{\sim}{\longrightarrow} A_n$ are easily obtained, as in (d). But the above derivation of the universal-coefficients spectral sequence does not work, because one seems to need to resolve the module variable both $\mathscr{S}$-relatively and K-relatively at the same time.

11. Appendix on acyclic models

Let $0 \longleftarrow C_{-1} \longleftarrow C_0 \longleftarrow C_1 \longleftarrow \cdots$ be a chain complex of functors $\mathbf{C} \longrightarrow \mathscr{A}$. (C_n) is **G**−**representable**, where **G** is a cotriple in **C**, if there are natural transformations $\vartheta_n \colon C_n \longrightarrow GC_n$ such that $\vartheta_n.\varepsilon C_n = C_n$ for all $n \geq 0$. (C_n) is **G-contractible** if the complex $(GC_n)_{n \geq -1}$ has a contracting homotopy (by natural transformations).

PROPOSITION [(11.1)]. [Barr & Beck (1966)] *Suppose that (C_n) is **G**-representabie, (K_n) is **G**-contractible, and $\varphi_{-1} \colon C_{-1} \longrightarrow K_{-1}$ is a given natural transformation, then φ_{-1} can be extended to a natural chain transformation $(\varphi_n) \colon (C_n) \longrightarrow (K_n)_{n \geq -1}$ by the inductive formula*

$$
\begin{array}{ccccc}
C_n & \overset{\vartheta_n}{\longrightarrow} & GC_n & \qquad GK_n & \overset{\varepsilon K_n}{\longrightarrow} & K_n \\
 & & \Big\downarrow{\scriptstyle G\partial} & \Big\uparrow{\scriptstyle h} & & \\
 & & GC_{n-1} & \xrightarrow[G\varphi_{n-1}]{} & GK_{n-1} &
\end{array}
$$

Any two extensions of φ_{-1} are naturally chain homotopic (we omit the formula).

In particular, if $C_{-1} = K_{-1}$, then there are natural chain equivalences $(C_n) \overset{\sim}{\underset{\longleftarrow}{\longrightarrow}} (K_n)$.

If $E\colon \mathbf{C} \longrightarrow \mathscr{A}$ is a functor with values in an additive category, then the standard chain complex

$$0 \longleftarrow E \longleftarrow GE \longleftarrow G^2 E \longleftarrow \cdots$$

is $\mathbf{G}$-representable and $\mathbf{G}$-contractible by virtue of $\partial G^n E\colon G^{n+1} E \longrightarrow G^{n+2} E$. Thus if

$$0 \longleftarrow E \longleftarrow E_0 \longleftarrow E_1 \longleftarrow \cdots$$

is any $\mathbf{G}$-representable chain complex of functors $\mathbf{C} \longrightarrow \mathscr{A}$, there exists a unique natural chain transformation $(\varphi_n)\colon (E_n) \longrightarrow (G^{n+1} E)$ such that $\varphi_{-1} = E$ (up to homotopy).

The proof is more of less contained in the statement. The term "$\mathbf{G}$-contractible" was not used in [Barr & Beck (1966)], the term "$\mathbf{G}$-acyclic" used there is reintroduced below with a different meaning.

The conclusions of (11.1) in practice are often too hard to establish and too strong to be relevant, At present all we need is homology isomorphism - a conclusion which is much weaker than chain equivalence. Thus it is convenient and reasonably satisfying to have the following weaker result available (as M. André has pointed out to us—see also [André (1967)]), that one can conclude a homology isomorphism $H(XE_*) \longrightarrow H(X, E)_{\mathbf{G}}$ from the information that the complex E_* is $\mathbf{G}$-representable as above and $\mathbf{G}$-acyclic merely in the sense that $H_n(XGE_*) = 0$ if $n > 0$, and $= XGE$ if $n = 0$. This observation greatly simplifies proofs of agreement between homology theories arising from standard complexes, such as those of [Barr & Beck (1966)].

PROPOSITION (11.2). *Let*

$$0 \longleftarrow E \longleftarrow E_0 \longleftarrow E_1 \longleftarrow \cdots$$

be a complex of functors $\mathbf{C} \longrightarrow \mathscr{A}$ *such that*

$$H_n(XGE_*) = \begin{cases} XGE, & n = 0 \\ 0 & n > 0, \end{cases}$$

and the $\mathbf{G}$*-homology groups*

$$H_p(X, E_q)_{\mathbf{G}} = \begin{cases} XE_q, & p = 0 \\ 0, & p > 0, \end{cases}$$

for all $X \in \mathbf{C}$, $q \geq 0$. *Then the spectral sequences obtained from the double complex*

$$(XG^{p+1} E_q)_{p, q \geq 0}$$

by filtering by levels $\leq p$ *and* $\leq q$ *both collapse, giving edge isomorphisms*

$$H_n(XE_*) \overset{\sim}{\longrightarrow} total\ H_n \qquad\qquad (p\ filtration)$$

$$H_n(X, E)_{\mathbf{G}} \overset{\sim}{\longrightarrow} total\ H_n \qquad\qquad (q\ filtration)$$

for all $n \geq 0$, hence natural isomorphisms $H_n(XE_*) \xrightarrow{\sim} H_n(X, E)_\mathbf{G}$.

In particular, **G**-representability of the complex (E_n) guarantees the second acyclicity condition, since the E_q are then retracts of the **G**-acyclic functors GE_q, $q \geq 0$.

There is an obvious overlap between these two propositions which we encountered in Theorem (8.1):

PROPOSITION (11.3). *Let $0 \longleftarrow E \longleftarrow E_0 \longleftarrow \cdots$ be a **G**-representable chain complex of functors $\mathbf{C} \longrightarrow \mathscr{A}$ and $(\varphi_n): (E_n) \longrightarrow (G^{n+1}E)$, $n \geq -1$, a chain transformation such that $\varphi_{-1} = E$ (see (11.1)). By **G**-representability, the acyclicity hypothesis*

$$H_p(X, E_q) = \begin{cases} XE_q, & p = 0 \\ 0, & p > 0, \end{cases}$$

is satisfied and the rows of the double complex $XG^{p+1}E_q$ have homology zero. We obtain a spectral sequence

$$H_p(X, H_q(\_, E_*))_G \Longrightarrow H_{p+q}(XE_*),$$

where the total homology is filtered by levels $\leq p$. The edge homomorphisms are

$$H_0(X, H_n(\_, E_*))_\mathbf{G} \xrightarrow{\lambda_\mathbf{G}} H_n(XE_*)$$

and the top map in the commutative diagram

$$H_n(XE_*) \longrightarrow H_n(X, H_0(\ , E_*))_\mathbf{G}$$

$$H_n(X\varphi_*) \searrow \qquad \downarrow$$

$$H_n(X, E)_\mathbf{G}$$

Finally suppose that

$$H_n(XGE_*) = \begin{cases} XGE, & n = 0 \\ 0 & n > 0. \end{cases}$$

The spectral sequence collapses, as $H_p(X, H_q(\_, E_*))_\mathbf{G} = 0$ if $q > 0$. The edge homomorphism $\lambda_\mathbf{G}$ is zero. The second edge homomorphism and the vertical map in the above triangle both become isomorphisms. Thus the homology isomorphism produced by (11.2) is actually induced by the chain map $\varphi_*: E_* \longrightarrow (G^{n+1}E)_{n \geq 0}$.

The proof is left to the reader.

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