TOPOLOGICAL BALLS

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Résumé. Cet article montre comment on peut utiliser la ≪construction de Chu≫ afin de simplifier la construction assez complexe de la catégorie $*$-autonome des boules qui sont réflexives et ζ-ζ^*-complètes donnée par le premier auteur dans les articles [Barr, 1976, 1979].

Abstract. This paper shows how the use of the "Chu construction" can simplify the rather complicated construction of the $*$-autonomous category of reflexive ζ-ζ^*-balls set up by the first author in the original papers and lecture notes on $*$-autonomous categories ([Barr, 1976, 1979]).

1. Introduction

By a *(topological) ball*, we mean the unit ball of a Banach space equipped with a second locally convex Hausdorff topology, coarser than that of the norm, in which the norm is lower semi-continuous. A morphism is a function, continuous in the second topology, that preserves the absolutely convex structure of the unit balls. We denote by $\mathscr{B}$ the category so defined. This category can also be viewed as (that is, it is equivalent to) the full subcategory of complex Saks spaces in the sense of Cooper [1987] generated by those Saks spaces whose underlying normed space is complete. For that reason, we often speak of functions that preserve the absolutely convex combinations as *linear*, even if that is not quite the correct word.

The first author constructed a full subcategory $\mathscr{R}$ of $\mathscr{B}$ which can be endowed with a closed symmetric monoidal structure and where every object is reflexive, that is, $\mathscr{R}$ is a $*$-*autonomous category.* The objects of $\mathscr{R}$ are the *reflexive* ζ-ζ^*-*balls* introduced in [Barr, 1976]. For the time being we do not need to know the definition of those balls. What is important, is that on the one hand the category $\mathscr{R}$ can be completed to a model of full linear logic ([Kleisli, *et al.* 1996]) and also yields an interesting group algebra for completely regular (for which T_0 suffices) topological groups ([Schläpfer, 1998], [Dorfeev, Kleisli, 1995]), and on the other hand, the construction of the category $\mathscr{R}$ suffers from complications of topology and completeness which are hard to describe and even harder to understand.

In a recent paper titled "$*$-autonomous categories, revisited" ([Barr, 1996]), the first author showed that a construction studied by P.-H. Chu in [Chu, 1979], now known as the "Chu construction", can basically replace complications such as those encountered in

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the original construction of the category $\mathscr{R}$. In this paper, we show how this is actually done. We will introduce an autonomous (that is, closed symmetric monoidal) category $\mathscr{A}$ with pullbacks, an object K in $\mathscr{A}$, and a factorization system $\mathscr{E}/\mathscr{M}$ in $\mathscr{A}$ such that the category $\mathscr{R}$ is equivalent to a full subcategory of the category $\mathrm{Chu}_{se}(\mathscr{A}, K)$ of $\mathscr{M}$-separated, $\mathscr{M}$-extensional Chu spaces of $\mathscr{A}$ with respect to K. We will describe that category in Section 4. Following a suggestion of V. Pratt, we will denote it by $\mathrm{chu}(\mathscr{A}, K)$. We will demonstrate that, under mild assumptions, this category is $*$-autonomous. Using this, we find a description of the category $\mathscr{R}$ which avoids many of the complications of the original construction and also gives a better insight into the topological considerations involved.

2. Some generalities

A ball will be called *discrete* if the topology is exactly that of the norm (which is automatically continuous with respect to itself). Notice that in a ball the norm is intrinsic, since in a ball B, we have

$$\|x\| = \inf\{\lambda > 0; x \in \lambda B\} \tag{$*$}$$

The full subcategory of discrete balls is denoted $\mathscr{B}_d$. This category is a symmetric closed monoidal, that is autonomous, category. The internal hom $[A, B]$ is the set of all linear (that is, absolutely convex) functions $A \longrightarrow B$. The tensor product is the completion of the linear tensor product with respect to the least cross-norm (the projective tensor product).

Let $\mathbf{D}$ denote the unit disk of the complex numbers. We will call a continuous linear map $B \longrightarrow \mathbf{D}$ a *functional* on B. A ball will be said to have a *weak topology* if no coarser topology allows the same set of functionals and will be said to have a *strong or Mackey topology* if any finer topology allows strictly more functionals. One of the things we will show is that given any ball, there is a both a coarsest and finest topology that has the same set of functionals.

We begin with the following proposition. The first three conditions are shown to be equivalent in [Cooper, 1987], I.3.1, and the fourth is a more categorical version.

2.1. PROPOSITION. *The following are equivalent for the unit ball B of a Banach space equipped with a Hausdorff topology given by a family Φ of seminorms.*

1. *B is a ball, that is the norm function is lower semicontinuous on B;*

2. *$\frac{1}{2}B$ is closed in B;*

3. *for any $b \in B$, $\|b\| = \sup_{\varphi\in\Phi} \varphi(b)$;*

4. *B is embedded isometrically and topologically in a product of discrete balls.*

PROOF. Because of the already noted equivalence of the first three, it is sufficient to show that $3 \Rightarrow 4 \Rightarrow 2$.

$3 \Rightarrow 4$. For each $\varphi \in \Phi$, let $B_\varphi = \ker\varphi$. This is a closed subball and φ is a continuous norm on B/B_φ, and extends to a continuous norm on its completion D_φ which is thereby a discrete ball. The topology of the seminorms embeds B topologically into $\prod_{\varphi\in\Phi} D_\varphi$ and 3 implies that the embedding is also an isometry.

$4 \Rightarrow 2$. This is immediate from the discrete case. ∎

From the equivalence of 1 and 2 above, we see:

2.2. COROLLARY. *Suppose B is the unit ball of a Banach space equipped with a compact Hausdorff topology given by a family of seminorms. Then B is a ball.*

There is a notable omission from [Barr, 1976, 1979], which we now fill.

2.3. PROPOSITION. *For any balls A and B, the set of continuous linear maps $A \longrightarrow B$ is complete in the operator norm.*

PROOF. Suppose the sequence of continuous linear functions $f_1, f_2, \ldots$ is a Cauchy sequence. Since B is complete, the sequence converges to a function f in the operator norm. We want to show that f is continuous. We must show that for any seminorm φ on B, the composite $\varphi \circ f$ is continuous on A. Suppose $\epsilon > 0$ is given. Choose n so that $\|f - f_n\| < \epsilon/2$ and then choose an open neighborhood $U \subseteq A$ of 0 so that $a \in U$ implies that $\varphi \circ f_n(a) < \epsilon/2$. Then one sees immediately, using that $\varphi \le \|-\|$, that $a \in U$ implies that $\varphi \circ f(a) < \epsilon$. ∎

3. Weak balls and Mackey balls

3.1. Let B' denote the set of all functionals on B. For the time being, we do not assign it a topology, except in two cases. If B is compact, we give B' the discrete topology and if B is discrete we give B' the topology of pointwise convergence, which is compact, since it is a closed subspace of $\mathbf{D}^B$. It is well known that these spaces are reflexive in the sense that the obvious evaluation maps $B \longrightarrow B''$ are topological and algebraic isomorphisms. See, for example, [Kleisli, Künzi], (2.12) and (4.2).

We want to demonstrate that given a ball B, there are both a weakest and a strongest topology on B which has the same set of functionals. In [Barr, 1996] this is shown for vector spaces over a discrete field and also for abelian groups, but the arguments depended on particularities of those categories and so we need another argument here. In fact, this argument is more general and would have worked for all three.

3.2. First we observe that there is a weakest topology with the same functionals. Namely, retopologize B with the weakest topology for which all the functionals are continuous. This amounts to embedding B into $\mathbf{D}^{B'}$. The retopologized ball clearly has the same set of functionals as the original. On the other hand, any topology on B that has the same set of functionals must map continuously into $\mathbf{D}^{B'}$ with the same image and hence into B. We call the weakest topology with the set B' of functionals $\sigma(B, B')$ although we often write σB for the ball B retopologized with $\sigma(B, B')$. If B already has the weak topology, we say that B is *weakly topologized.*

Since every functional in B' is continuous on B, the topology on σB is weaker than that of B, that is $B \longrightarrow \sigma B$ is continuous. This suggests the following:

3.3. Theorem. *The weakly topologized balls form a reflective subcategory $\mathscr{S}$ of $\mathscr{B}$ with reflector σ. The weak topology on a ball is the topology of pointwise convergence on its dual; a continuous seminorm is the absolute value of the evaluation on an element of the dual.*

Proof. Since σB has the same functionals as B, it is evident that σ is idempotent. Thus, given $B \longrightarrow A$ with A weakly topologized, then we have $\sigma B \longrightarrow \sigma A = A$. This shows the adjunction. The remainder is implicit in the preceding discussion. ∎

3.4. Sums. The next thing we have to do is investigate sums in the category of balls. We observe that in any ball B, it makes sense to write $\sum b_i$ for any collection, possibly infinite, of elements, so long as $\sum \|b_i\| \le 1$. For suppose that b is a non-zero element of B. From the formula $(*)$ for $\|b\|$ at the beginning of Section 2, it follows that for any $\lambda > \|b\|$ there is an element b_λ with $b = \lambda b_\lambda$. If we restrict to a sequence of λ that converges to $\|b\|$, the resultant sequence of b_λ is evidently a Cauchy sequence that converges in the norm to an element we may as well denote $b/\|b\|$ such that $b = \|b\|(b/\|b\|)$. Then $\sum b_i = \sum \|b_i\|(b_i/\|b_i\|)$ is a totally convex linear combination.

3.5. Proposition. *Let $\{B_i\}$ be a collection of balls. Let B be the set of all formal sums $\sum b_i$ such that $\sum \|b_i\| \le 1$ with the latter sum as norm. Topologize it by the finest topology such that each inclusion $B_i \longrightarrow B$ is continuous. Then this is the sum in the category $\mathscr{B}$.*

Proof. Suppose, for each $i \in I$ there is given a morphism $f_i \colon B_i \longrightarrow C$. Define $f \colon B \longrightarrow C$ by $f(\sum b_i) = \sum f_i(b_i)$. This sum is well defined since $\sum \|f_i(b_i)\| \le \sum \|b_i\| \le 1$. It is clear that $f|B_i = f_i$ and that f is unique with that property. If we give B the weak topology for f, then the topology will restrict to a topology on each B_i that makes f_i continuous and is therefore coarser than the given topology on B_i. Thus the weak topology defined on B by f is coarser than the strong topology defined by all the f_i and hence f is continuous in that latter topology. ∎

3.6. Proposition. *Let $\{B_i\}$, $i \in I$ be a collection of balls. The natural map $\sum B_i' \longrightarrow (\prod B_i)'$ is a bijection.*

Proof. There is, for each $i \in I$, a product projection $\prod B_i \longrightarrow B_i$, that dualizes to a map $B_i' \longrightarrow (\prod B_i)'$ which gives a map $\sum B_i' \longrightarrow (\prod B_i)'$. It is easy to see, using elements of the product that are 0 in every coordinate but one, that this map is injective. Now let β be a continuous functional on $\prod B_i$. Let β_i be the restriction of β to B_i. We have to show that $\sum \|\beta_i\| \le 1$ (in particular that at most countably many are non-zero). If not, $\sum \|\beta_i\| > 1+\epsilon$ for some $\epsilon > 0$ and then there is a finite set of indices, say $1, \ldots, k$ for which $\sum_{i=1}^{k} \|\beta_i\| > 1 + \epsilon/2$. Let $b_i \in B_i$ be an element for which $\beta_i(b_i) > \|\beta_i\| - \epsilon/4k$ for $i = 1, \ldots k$ and $b_i = 0$ otherwise. Then for $b = \{b_i\}$,

$$\beta(b) = \sum_{i=1}^{k} \beta_i(b_i) > \sum_{i=1}^{k} (\|\beta_i\| - \epsilon/4k) > 1 + \epsilon/2 - \epsilon/4 = 1 + \epsilon/4$$

which is impossible. ∎

At this point, we must assign a topology to B'. Although there is some choice in the matter, we will topologize it by uniform convergence on compact subballs. In other words, we topologize the dual of a compact ball discretely and if $\{C_i\}$ ranges over the compact subballs of B, then B' is topologized as a subspace of $\prod C_i'$. This obviously extends the already given topology on the duals of compact balls. It also extends the one on discrete balls, which are topologized by pointwise convergence. The reason is the well-known fact that a Banach space with compact unit ball is finite dimensional. Thus the compact subballs are finite dimensional and pointwise convergence there is the same as pointwise convergence on a finite basis.

3.7. Proposition. *For any ball B, the natural evaluation map $B \longrightarrow B''$ is an open, but not necessarily continuous, bijection.*

Proof. The fact that it is a bijection is found in [Barr, 1979], IV (3.18). The topology on B is completely determined by maps $B \longrightarrow D$, with D discrete and any such map gives a map $B'' \longrightarrow D'' \cong D$. ∎

Now say that a map $B \longrightarrow A$ is *weakly continuous* if the composite $B \longrightarrow A \longrightarrow \sigma A$ is continuous. We now define τB as the ball B retopologized with the weak topology for all weakly continuous maps out of B.

As usual, one can readily show that $B' \cong B'''$ so that $B \longrightarrow B''$ is weakly continuous.

3.8. Proposition. *The identity map $\tau B \longrightarrow B$ is continuous, τB has the same continuous functionals as B and τB has the largest topology for which this is true.*

Proof. Every weakly continuous morphism out of B is, by definition, continuous on τB. Since the identity map is weakly continuous, $\tau B \longrightarrow B$ is continuous. For any ball B_1 with the same point set and a topology finer than that of τB, the identity $B \longrightarrow B_1$ is not even weakly continuous, which means there is some continuous functional on B_1 that is not continuous on B. Thus any topology finer than that of τB has more functionals than B.

To finish, we have to show that every functional continuous on τB is continuous on B. Since there is only a set of topologies on B, we can find a set of weakly continuous $B \longrightarrow B_i$ such that τB has the weak topology for that set of arrows. This evidently means that

$$\begin{tikzcd} \tau B \arrow[r] \arrow[d] & \prod B_i \arrow[d] \\ \sigma B \arrow[r] & \prod \sigma B_i \end{tikzcd}$$

is a pullback. Since one of the possibilities for B_i is the identity of B, one easily sees that $\sigma B \longrightarrow \prod \sigma B_i$ and hence $\tau B \longrightarrow \prod B_i$ are isometric embeddings. Now let φ be a continuous functional on τB. The fact that $\mathbf{D}$ is an injective object ([Barr, 1979], 3.17) implies that φ extends to a continuous functional ψ on $\prod B_i$. From the preceding proposition, it follows that the dual space of $\prod B_i$ is $\sum B_i'$, which maps bijectively to $\sum \sigma B_i' = (\prod B_i)'$. Thus ψ remains continuous on $\prod \sigma B_i$ and then restricts to a continuous functional φ on B. ∎

By a *Mackey ball*, we mean a ball B for which the identity function $\tau B \longrightarrow B$ is an isomorphism of balls.

3.9. Remark. In [Barr, forthcoming], the same argument will be used to show the existence of the Mackey topology—also the finest with a given set of continuous functionals—for the case of locally convex topological vector spaces. This is, of course, a classical result, but our categorical proof is much simpler than the one found in the standard literature. In the locally convex case, Proposition 3.5 has to be replaced by the theorem that identifies the dual space of a product with the ordinary algebraic direct sum of the dual spaces, see [Schaeffer, 1971], IV.4.3.

3.10. Seminorms on τB. We wish to characterize the seminorms on τB. All seminorms on B arise as composites $B \longrightarrow D \xrightarrow{\|-\|} \mathbf{D}$ where $B \longrightarrow D$ is a continuous arrow and D is discrete. We can, by replacing the map by the completion of its epimorphic image, suppose the map is an epimorphism, which means that $D' \longrightarrow B'$ is injective (but not generally an isometry). Since D is discrete, D' is compact. Seminorms on τB arise in the same way from weakly continuous $B \longrightarrow D$ followed by the norm on D. Moreover, even a weakly continuous $B \longrightarrow D$ induces $D' \longrightarrow B'$ by the definition of weak continuity.

3.11. Proposition. *The composite* $B \xrightarrow{f} D \xrightarrow{\|-\|} \mathbf{D}$ *is given by*

$$p(b) = \sup_{\varphi \in D'} |f'(\varphi)(b)|$$

Proof. The definition of f' is that $f'(\varphi)(b) = \varphi(f(b))$. Given that for any Banach ball A and $a \in A$, we have that

$$\|a\| = \sup_{\varphi \in A'} |\varphi(a)|$$

and the conclusion follows. ∎

Thus every seminorm on τB is given as such a sup taken over some compact subball of B'. Conversely, if C is a compact subball of B', then we have $B'' \longrightarrow C'$ and C' is discrete. Since $B \longrightarrow B''$ is weakly continuous (they have the same elements and the same functionals), it follows that the seminorm $B \longrightarrow C' \xrightarrow{\|-\|} \mathbf{D}$ is a seminorm on τB. Thus we have proved,

3.12. Proposition. *Every seminorm on the ball τB has the form $\sup_{\varphi \in C} |\varphi b|$ for a weakly compact subball $C \subseteq B'$.*

In the case of topological vector spaces, the Mackey topology is described as that of uniform convergence on subsets of the dual space that are compact in the weak topology. To make the analogy stronger, the following result implies that the compact subballs are the same for any compatible topology.

3.13. Corollary. *If B is any ball, then any compact subball of σB is also compact as a subball of B.*

Proof. If C is a compact subball of σB, then the continuous inclusion gives an embedding $C = \tau C \longrightarrow \tau\sigma B = \tau B$ and the topology of τB is weaker than that of B. ■

The results on Mackey balls can be summarized as follows.

3.14. Theorem. *The Mackey balls form a coreflective subcategory $\mathscr{T}$ of $\mathscr{B}$ with coreflector τ. The Mackey topology on a ball is the topology of uniform convergence on compact subballs of its dual; equivalently, a continuous seminorm is the supremum of absolute value of the evaluation on a compact subball of the dual.*

4. The Chu category

4.1. Definition. We recall the definition of a Chu category $\mathrm{Chu}(\mathscr{A}, \perp)$ for an autonomous category $\mathscr{A}$ and an object $\perp$. An object of this category is a pair (A_1, A_2) equipped with an arrow $A_1 \otimes A_2 \longrightarrow \perp$, called a pairing. A morphism $(A_1, A_2) \longrightarrow (B_1, B_2)$ is a pair $f_1 \colon A_1 \longrightarrow B_1$ and $f_2 \colon B_2 \longrightarrow A_2$ (note the direction of the second arrow) such that the square

$$\begin{tikzcd}
A_1 \otimes B_2 \arrow[r, "A_1 \otimes f_2"] \arrow[d, "f_1 \otimes B_2"'] & A_1 \otimes A_2 \arrow[d] \\
B_1 \otimes B_2 \arrow[r] & \perp
\end{tikzcd}$$

commutes, the other two arrows being the respective pairings. This category is obviously self dual; the duality reverses the components. What is interesting is that, provided $\mathscr{A}$ has pullbacks, it is still autonomous, now $*$-autonomous ([Chu, 1979]).

In the case at hand, we take for $\mathscr{A}$ the category $\mathscr{B}_d$ of discrete balls and for $\perp$ the unit ball $\mathbf{D}$ of the complex numbers. It is evident that $\mathscr{B}_d$ is a complete category and, in particular, has pullbacks.

4.2. THE FACTORIZATION SYSTEM ON $\mathscr{B}_d$. We let $\mathscr{E}$ denote the class of epimorphisms in $\mathscr{B}_d$ and $\mathscr{M}$ the class of closed isometric embeddings. We claim that $\mathscr{E}$ and $\mathscr{M}$ constitute a factorization system that satisfies the conditions of [Barr, 1998], 1.3. These conditions are

FS–1. Every arrow in $\mathscr{E}$ is an epimorphism;

FS–2. if $m \in \mathscr{M}$, then for any object A of $\mathscr{A}$, the induced $A \multimap m$ is in $\mathscr{M}$.

First we have to show it is a factorization system. To do this, we factor each arrow $A \xrightarrow{f} B$ as $A \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow B$. Here A_1 is the actual image of f,

$$A_2 = \{b \in B; \lambda b \in A_1 \text{ for some } \lambda \le 1\}$$

and A_3 is the closure of A_2. It is clear that each of the three maps is an epimorphism and that A_3 has a closed isometric embedding into B. We claim that that embedding is a regular, hence extremal, epimorphism and hence the diagonal fill-in condition is satisfied. The reason is that A_3 is a closed isometrically embedded subball and so we can form the quotient B/A_3, which will be Hausdorff and a ball. The kernel of $B \longrightarrow B/A_3$ is A_3.

Since $\mathscr{E}$ is exactly the epimorphisms, FS–1 is certainly satisfied. FS–2 is easy to check since internal homs preserve kernels.

Using Theorem 3.3 of [Barr, 1998] we conclude that,

4.3. THEOREM. *The category* $\mathrm{chu}(\mathscr{B}_d, \mathbf{D})$ *of* $\mathscr{M}$*-separated and* $\mathscr{M}$*-extensional Chu spaces is* $*$*-autonomous.*

4.4. COMPARISON BETWEEN CHU AND $\mathscr{B}$. Define $F\colon \mathscr{B} \longrightarrow \mathrm{chu}(\mathscr{B}_d, \mathbf{D}))$ by $FB = (|B|, |B'|)$ where $|B|$ is the discrete space underlying B. This is clearly extensional since $|B'|$ is normed as the dual of B. It is also separated; it follows from [Barr, 1979], IV (3.14), that for each $b \in B$, there is a continuous functional φ such that $\varphi(b)$ is arbitrary close to $\|b\|$.

4.5. THEOREM. *The functor* F *has both a left adjoint* L *and a right adjoint* R*. Moreover* R *induces an equivalence between* $\mathrm{chu}(\mathscr{B}_d, \mathbf{D})$ *and* $\mathscr{S}$ *and* L *an equivalence between* $\mathrm{chu}(\mathscr{B}_d, \mathbf{D})$ *and* $\mathscr{T}$*.*

PROOF. Define $R(B_1, B_2)$ as the space B_1 topologized by the weak topology from B_2. This means that $R(B_1, B_2)$ is embedded topologically in $\mathbf{D}^{B_2}$. The embedding is also isometric since both halves of $B_1 \rightarrowtail [B_2, \mathbf{D}] \longrightarrow \mathbf{D}^{B_2}$ are, the latter by the Hahn-Banach theorem. Thus $R(B_1, B_2)$ is a ball in our sense. We claim the functionals on

$R(B_1, B_2)$ are all represented by elements of B_2. For a ball B, let $B^\perp$ represent the weak dual of B. From $B_1 \rightarrowtail {B_2}^\perp$, we get

$$B_2 \longrightarrow {B_2}^{\perp\perp} \longrightarrow\!\!\!\!\!\rightarrow {B_1}^\perp$$

The first arrow is a bijection since weak balls are reflexive and the second is a surjection by [Barr, 1979], IV (3.17). Since the composite is an injection it follows that it is a bijection.

Define $L(B_1, B_2) = \tau R(B_1, B_2)$. Thus $R(B_1, B_2)$ is B_1 with the weakest topology for which B_2 is the dual space and $L(B_1, B_2)$ is B_1 with the strongest such topology. From this observation, the conclusion is obvious. It is also obvious that both FR and FL are equivalent to the identity, so that R and L are full and faithful and are distinct embeddings of $\mathrm{chu}(\mathscr{B}_d, \mathbf{D})$ into $\mathscr{B}$. It is clear that $\mathscr{S}$ is the image of R and $\mathscr{T}$ is the image of L. ∎

5. The autonomous category $\mathscr{R}$

We now concentrate our attention on the category $\mathscr{T}$ of Mackey balls. It is a $*$-autonomous category and we will denote the internal hom by $\multimap$, the tensor product by $\otimes$ and the dual of B by $B' = B \multimap \mathbf{D}$. Recall that the topology on B is that of uniform convergence on the compact subballs of B' (3.12 of Section 3.) This leads to the question, when is this the compact/open topology, that is the topology of uniform convergence on compact sub*sets*?

This will surely be so if every compact subset generates a compact subball. And there is a natural class of objects for which this happens. It is known that a compact subset always generates a totally bounded subball. This leads to the concept of ζ-completeness introduced in [Barr, 1979], also in connection with $*$-autonomous categories. We say that a ball is a ζ-*complete* ball, or simply a ζ-ball, if every closed totally bounded subball is compact.

5.1. Theorem. *If B is a ζ-complete ball, then the topology on B' of uniform convergence on compact subballs coincides with the compact/open topology.*

Proof. The compact/open topology ω is the topology of uniform convergence on compact subsets. Thus the Mackey topology τ is coarser than ω. To show the opposite inequality we have to find, for each compact subset $K \subseteq B$, a compact subball $C \subseteq B$ containing K so that for all $b \in B$,

$$\sup_{\beta\in K} \beta(b) \le \sup_{\beta\in C} \beta(b)$$

Define C as the closure of the absolute convex hull of K. It is shown in [Bourbaki, 1953], Proposition 2 of II.4 that C so defined is totally bounded and hence, in a ζ-complete ball, compact. It follows from Corollary 2.2 that C is a subball of B. ∎

5.2. In [Barr, 1976], Proposition 2.1 we find he following characterization of ζ-balls: B is a ζ-ball if and only if any morphism from a dense subball of a compact ball into B can be extended to the whole compact ball. In particular, a complete ball is ζ-complete. Let us denote by $\zeta\mathscr{B}$ the full subcategory of $\mathscr{B}$ consisting of the ζ-complete balls, then the inclusion $\zeta\mathscr{B} \longrightarrow \mathscr{B}$ has a left adjoint we denote $\zeta: \mathscr{B} \longrightarrow \zeta\mathscr{B}$. Namely, embed $B \subseteq \prod D_i$. This product is complete, so let ζB be the intersection of all ζ-complete subballs of the product that include B. This last is the object function of the ζ-completion. See [Barr, 1979], III (1.4) (where the uniform completion is used instead of the product) for details. Note that the characterization we have just given of ζ-balls makes it evident that this intersection is still a ζ-ball. We have the following proposition.

5.3. Proposition. *For any ball B*

1. *if B is a ζ-ball, then the associated Mackey ball τB is also a ζ-ball;*

2. *if B is a Mackey ball, then so is ζB.*

Proof. Let B be a ζ-ball, C_0 a dense subball of a compact ball C and $\rho_0: C_0 \longrightarrow \tau B$ a continuous morphism. Since B is a ζ-ball, there is a continuous morphism $\rho: C \longrightarrow B$ for which the diagram

$$\begin{tikzcd} C_0 \arrow[r] \arrow[d, "\rho_0"'] & C \arrow[d, "\rho"] \\ \tau B \arrow[r] & B \end{tikzcd}$$

commutes. We have to show that f considered as a morphism $C \longrightarrow \tau B$ is continuous. By hypothesis, for every weakly compact subball $K \subseteq (\tau B)'$ there is a continuous seminorm φ on C_0 and a positive constant c such that $\sup_{y\in K} |y(\rho_0 x)| \le c\varphi x$ for all $x \in C_0$. Let ψ be the unique continuous seminorm on C that extends the seminorm φ on C_0. Since C_0 is dense in C, for every $x \in C$ there is a Cauchy net (x_α) in C_0 that converges to x. Hence $\psi x = \lim \varphi x_\alpha$ and, since f is weakly continuous, we have $|y(fx)| = \lim |y(\rho_0 x_\alpha)|$ for all $y \in B'$. In other words, for every $\epsilon > 0$, there are indices α_0 and $\alpha(y)$ such that $|\varphi x_\alpha - \psi x| < \epsilon$ for all $\alpha > \alpha_0$ and $|y(fx)| - |y(f_0 x_\alpha)| < \epsilon$ for all $\alpha > \alpha(y)$. Hence

$$|y(fx)| < |y(f_0 x_\alpha)| + \epsilon \le c\varphi(x_\alpha) + \epsilon \le c\psi x + 2\epsilon$$

for all $\alpha \ge \max(\alpha_0, \alpha(y))$. Therefore, $|y(fx)| < c\psi x$ for all all $y \in K$ and $x \in C$, so that $\sup_{y\in K} |y(fx)| \le c\psi x$. This means that $f: C \longrightarrow \tau B$ is continuous. ∎

It follows that the full subcategory $\zeta\mathscr{T}$ of $\mathscr{T}$ given by the ζ-complete Mackey balls is also a reflective subcategory of $\mathscr{T}$ with the reflector given by the restriction of ζ. The objects of $\zeta\mathscr{T}$ are the reflexive ζ-balls of [Barr, 1979]. The reason is that it is just the Mackey balls that are reflexive for the strong topology on the dual ball. The dual category $\zeta^*\mathscr{T}$ is a full coreflective subcategory of $\mathscr{T}$ with coreflector given by $\zeta^* B = (\zeta B')'$. We denote by $\mathscr{R}$ the category $\zeta\mathscr{T} \cap \zeta^*\mathscr{T}$ of reflexive ζ-ζ^* balls of [Barr, 1979].

5.4. Theorem. *The category $\mathscr{R}$ is $*$-autonomous with tensor product $A \boxtimes B = \zeta(A \otimes B)$, internal hom $A \multimap\!\!\square\, B = \zeta^*(A \multimap B)$ and $\mathbf{D}$ as unit and dualizing object.*

Proof. By Theorem 2.3 of [Barr, 1998] we have to verify

1. if B is in $\zeta^*\mathscr{T}$, then so is ζB;
2. if A is in $\zeta^*\mathscr{T}$ and B is in $\zeta\mathscr{T}$, then $A \multimap B$ is in $\zeta\mathscr{T}$.

These are found in [Barr, 1976], Propositions 2.6 and 3.3. ∎

Summing up, we have established that the category $\mathscr{R}$ of reflexive ζ-ζ^* balls is $*$-autonomous in a transparent way by using "Chu space techniques" made available in [Barr, 1998].

6. Is Mackey always δ?

The original paper, as published contained at this point a section claiming a negative answer to this question. The question is now open. Here is the first paragraph of the original, which explains the significance of the question.

> Originally, it seemed possible that the $*$-autonomous category of δ-balls was the simply $\operatorname{chu}(\mathscr{B}_d, \mathbf{D})$ (recall that $\mathscr{B}_d$ is the category of discrete balls). The δ-balls are a full subcategory of the Mackey balls, which is equivalent to $\operatorname{chu}(\mathscr{B}_d, \mathbf{D})$ and what this claim really is is that it is the whole category. The purpose of this section is to show that there is a Mackey ball that is not ζ-complete. This leaves open the question of whether it is possible that the category of δ-balls is a chu category.

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