Journal of Pure and Applied Algebra 17 (1980) 1–24

ATOMIC TOPOSES*

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Communicated by F.W. Lawvere
Received 23 January 1979

1. Principal results

Before giving the statements, we require a definition. An *atomic site* is a category equipped with a topology in which every cover is non-empty and conversely, every non-empty sieve is a cover. In order that a category admit such a topology – evidently unique – it is necessary and sufficient that any pair of maps with a common codomain be completable to a commutative square (terminating in that codomain). This topology is subcanonical iff every morphism in the category is a regular epimorphism in the sense of being the common coequalizer of all pairs of maps that it coequalizes. The condition that the covers be non-empty is non-trivial for a strict initial object is covered, in any sub-canonical topology by the empty sieve.

Theorem A. *Let $\mathscr{E}$ be a Grothendieck topos, $\Gamma: \mathscr{E} \to \mathscr{S}et$ be the global sections functor with left adjoint Δ. Then the following are equivalent*:

(i) *$\mathscr{E}$ is the category of sheaves for an atomic site,*
(ii) *Δ is logical,*
(iii) *the subobject lattice of every object of $\mathscr{E}$ is a complete atomic boolean algebra.*

The first condition above is of course appropriate only for a Grothendieck topos. The second can be asked of any geometric morphism of toposes as can a suitable modification of the third. When this is done, we will see that the last two conditions remain equivalent.

* The first author would like to thank the National Research Council of Canada and the Ministère de l'Education du Québec and the second the National Science Foundation of the United States for their support of this research.

If E is an object of $\mathscr{E}$ (or of any category) the statement that the subobject lattice of E is a complete atomic boolean algebra is equivalent to the existence of a set ΛE – the atoms of E – and an order isomorphism between the subobject lattice of E and the boolean algebra $2^{\Lambda E}$. Since $\mathscr{E}$ is a topos, the subobject lattice of E is $\mathrm{Hom}(E, \Omega) \cong \mathrm{Hom}(1, \Omega^E) = \Gamma(\Omega^E)$, where Ω is the subobject classifier of $\mathscr{E}$. Thus the third condition of the theorem can be replaced by

(iii)′ There is an object function $\Lambda : \mathscr{E} \to \mathscr{S}et$ such that for any $E \in \mathscr{E}$,

$$\Gamma(\Omega^E) \cong 2^{\Lambda E}$$

as partially ordered sets.

We are now ready to state the generalization of Theorem A. Here and elsewhere, we let Ω denote the subobject classifier of any topos (except when it is known to be boolean, in which case we use 2).

Theorem B. *Let $(\Delta, \Gamma) : \mathscr{E} \to \mathscr{B}$ be a geometric morphism between toposes. Then Δ is logical iff there is an object function $\Lambda : \mathscr{E} \to \mathscr{B}$ such that for any $E \in \mathscr{E}$ the partially ordered set objects $\Gamma(\Omega^E)$ and $\Omega^{\Lambda E}$ of B are isomorphic. In that case Λ can be extended to a functor left adjoint to Δ.*

In addition we study various properties and further characterizations of local homeomorphisms – those geometric morphisms (Δ, Γ) for which Δ is logical. We mention some examples which seem interesting and important.

For generalities about toposes we refer to [6].

2. Proof of Theorem B

Suppose that Δ is logical. A theorem of Mikkelsen's (but the proof sketched below is due to Paré) asserts that a logical functor has a left adjoint iff it has a right adjoint. Since Δ has a right adjoint Γ, it also has a left adjoint Λ. Paré's argument is based on the diagram

$$\begin{array}{ccc} \mathscr{E}^{\mathrm{op}} & \xleftarrow{\ \Delta^{\mathrm{op}}\ } & \mathscr{B}^{\mathrm{op}} \\ \Omega^{(\ )} \uparrow \downarrow \Omega^{(\ )} & & \Omega^{(\ )} \uparrow \downarrow \Omega^{(\ )} \\ \mathscr{E} & \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} & \mathscr{B} \end{array}$$

in which the downward arrows are tripleable and the fact that Δ is logical means that the square going from lower right to upper left commutes. Since Δ is left exact, Δ^{op} preserves (regular) epis. Then a theorem of Butler [2] implies that Δ^{op} has a right adjoint which we shall call Λ^{op}, and that

$$\Omega^{(\ )} \cdot \Lambda^{\mathrm{op}} \cong \Gamma \cdot \Omega^{(\ )}.$$

This course means that Λ is left adjoint to D and that for any E in $\mathscr{E}$,

$$\Omega^{\Lambda E} \cong \Gamma(\Omega^E).$$

Proposition 1. *The isomorphism above is a semilattice isomorphism, i.e., an isomorphism of inf semilattice objects (hence of partially ordered objects and hence of complete heyting algebra objects) of* $\mathscr{B}$.

Proof. To say that $\Omega^{\Lambda E}$ and $\Gamma(\Omega^E)$ are semilattice isomorphic is to assert that for any $B \in \mathscr{B}$ the isomorphism

$$\operatorname{Hom}(B, \Omega^{\Lambda E}) \cong \operatorname{Hom}(B, \Gamma(\Omega^E))$$

given by the adjointness

$$\Delta^{\mathrm{op}} \cdot \Omega^{(\ )} \dashv \Omega^{(\ )} \cdot \Delta,$$

together with the isomorphism of $\Delta\Omega \cong \Omega$, is a semilattice isomorphism. Since a semilattice is a model of an equational theory, we need only verify that the isomorphism is in one direction a map which preserves the semilattice structure. The isomorphism is the composite

$$\begin{aligned}
&\operatorname{Hom}(B, \Omega^{\Lambda E}) \xrightarrow{1} \operatorname{Hom}(B \times \Lambda E, \Omega) \xrightarrow{2} \operatorname{Hom}(\Delta(B \times \Lambda E), \Delta\Omega) \\
&\xrightarrow{3} \operatorname{Hom}(\Delta B \times \Delta\Lambda E, \Delta\Omega) \xrightarrow{4} \operatorname{Hom}(\Delta B \times \Delta\Lambda E, \Omega) \xrightarrow{5} \operatorname{Hom}(\Delta B \times E, \Omega) \\
&\xrightarrow{6} \operatorname{Hom}(\Delta B, \Omega^E) \xrightarrow{7} \operatorname{Hom}(\Gamma\Delta B, \Gamma(\Omega^E)) \xrightarrow{8} \operatorname{Hom}(B, \Gamma(\Omega^E)).
\end{aligned}$$

The maps labeled 1 and 6 are semilattice isomorphisms; the structures on $\Omega^{\Lambda E}$, resp. Ω^E, are defined (from those of the Ω's) by requiring that those maps be semilattice isomorphisms.

The maps labeled 2 and 7 are instances of a general principle. If $F: \mathscr{X} \to \mathscr{Y}$ is a finite-product-preserving functor and $X \in \mathscr{X}$ is a model of the finitary theory $\mathscr{T}h$, then FX is canonically a model of the same theory in $\mathscr{Y}$ in such a way that the function

$$\text{apply } F: \operatorname{Hom}(X'; X) \to \operatorname{Hom}(FX', FX)$$

is a $\mathscr{T}h$ homomorphism in $\mathscr{S}et$. In the present instance, the semilattice structures on $\Delta\Omega$ and $\Gamma(\Omega^E)$ are induced from those on Ω and Ω^E, respectively, by this canonical process applied to Δ and Γ. The maps labeled 3, 5 and 8 are instances of the facts that a morphism $X' \to X''$ in a category $\mathscr{X}$ induces a $\mathscr{T}h$ homomorphism

$$\operatorname{Hom}(X'', X) \to \operatorname{Hom}(X', X)$$

in $\mathscr{S}et$ under the conditions described above. As for 4, we leave it as an exercise to the reader to show that the map $d: \Delta\Omega \to \Omega$ which is the characteristic map of Δ(true) preserves finite intersections and hence is a morphism of semilattice objects. Thus the map

$$\Omega^{\Lambda E} \to \Gamma(\Omega^E)$$

preserves the semilattice structure and in particular the partial order. Thus it is an equivalence of partial orders.

This completes the proof of the "only if" part of Theorem B. To go the other way, recall that a regular category $\mathscr{X}$ is one in which pullbacks exist and the pullback of a regular epimorphism is regular. In such a category every map factors uniquely as a regular epimorphism followed by a monomorphism; a map which is both is an isomorphism (see [1, I.2]). We let

$$\mathrm{sub}:\mathscr{X}^{\mathrm{op}} \to \mathscr{S}emilattice$$

denote the functor which assigns to each object of X the inf semilattice of its subobjects. If $f:X \to Y$ is a morphism and $Y_0 \rightarrowtail Y$ is a subobject, $\mathrm{sub}(f)(Y_0)$ is the pullback $Y_0 \times_y X$. We conform to accepted usage and write

$$f^*:\mathrm{sub}\ Y \to \mathrm{sub}\ X$$

instead of $\mathrm{sub}(f)$.

Theorem C. *Let $\mathscr{X}$ and $\mathscr{W}$ be regular categories and $\Phi:\mathscr{X} \to \mathscr{W}$ be a finite product preserving functor. Let $X \in \mathscr{X}$ and $W \in \mathscr{W}$ be objects such that the functors*

$$\mathrm{sub}(X \times -), \mathrm{sub}(W \times \Phi\ -):\mathscr{X}^{\mathrm{op}} \to \mathscr{S}emilattice$$

are naturally equivalent. Then the functors

$$\mathrm{Hom}(X, -), \mathrm{Hom}(W, \phi\ -):\mathscr{X} \to \mathscr{S}et$$

are also naturally equivalent (maps identified with their graphs).

What has to be shown is that for $Y \in \mathscr{X}$, a subobject of $X \times Y$ is the graph of a map $X \to Y$ iff the corresponding subobject of $W \times \Phi Y$ is the graph of a map $W \to \Phi Y$. This will follow if we can characterize those subobjects which are graphs of maps in terms of the semilattice structure and functoriality of $\mathrm{sub}(X \times -)$ and $\mathrm{sub}(W \times \Phi\ -)$. We begin with

Proposition 2. *For $f: Y \to Z$ in $\mathscr{X}$, let*

$$\exists f: \mathrm{sub}\ Y \to \mathrm{sub}\ Z$$

assign to $Y_0 \rightarrowtail Y$ the image of the composite

$$Y_0 \rightarrowtail Y \to Z.$$

Then $\exists f$ is left adjoint to f^*.

Proof. Standard.

Proposition 3. *Let Y, Z be in the regular category $\mathscr{X}$. Let R be a subobject of $Y \times Z$. Let $p: Y \times Z \to Z$, $q_1, q_2: Y \times Z \times Z \to Y \times Z$ be projections and $d: X \times Z \rightarrowtail Y \times Z \times Z$ be*

$Y \times$ diagonal. Then R is the graph of a map $Y \xrightarrow{f} Z$ iff

(i) $\exists p(R) = Y$,

(ii) $q_1^*(R) \cap q_2^*(R) = \exists d(R)$.

Proof. In the category of sets the first condition describes the fact that R is defined everywhere in Y; the second that it is single-valued. More generally a subobject $R \rightarrowtail Y \times Z$ is the graph of a map $Y \to Z$ iff the composite

$$R \to Y \times Z \to Y$$

is an isomorphism. The data of the proposition are all preserved by exact functors. They are reflected by exact functors that reflect isomorphisms. Using the metatheorem of [1, III.6], we see that it is sufficient to prove it in $\mathcal{S}ets$.

This result can be summarized by saying that the set diagram:

$$\begin{array}{ccccc} \mathrm{Hom}(Y,Z) & \longrightarrow & \mathrm{sub}(Y \times Z) & \begin{array}{c} \to 1 \xrightarrow{\ulcorner Y \urcorner} \\ \xrightarrow[\exists p]{} \end{array} & \mathrm{sub}\, Y \\ & & \exists d \Big\downarrow \; \Big\downarrow q_1^* \wedge q_2^* & & \\ & & \mathrm{sub}(Y \times Z \times Z) & & \end{array}$$

is a limit. Here $\ulcorner Y \urcorner$ is the name of the largest subobject of Y.

Proposition 4. *Each of the following diagrams commutes.*

$$\begin{array}{ccccc} \mathrm{sub}(X \times Y) & \longrightarrow & 1 & \xrightarrow{\ulcorner X \urcorner} & \mathrm{sub}(X) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{sub}(W \times \Phi Y) & \longrightarrow & 1 & \xrightarrow{\ulcorner W \urcorner} & \mathrm{sub}(W) \end{array}$$

$$\begin{array}{ccc} \mathrm{sub}(X \times Y) & \xrightarrow{\exists p} & \mathrm{sub}(X) \\ \downarrow & & \downarrow \\ \mathrm{sub}(W \times \Phi Y) & \xrightarrow{\exists p} & \mathrm{sub}(W) \end{array}$$

$$\begin{array}{ccc} \mathrm{sub}(X \times Y) & \xrightarrow{q_i^*} & \mathrm{sub}(X \times Y \times Y) \\ \downarrow & & \downarrow \\ \mathrm{sub}(W \times \Phi Y) & \xrightarrow{q_i^*} & \mathrm{sub}(W \times \Phi Y \times \Phi Y) \end{array}$$

$$\begin{array}{ccc} \mathrm{sub}(X \times Y) & \xrightarrow{\exists d} & \mathrm{sub}(X \times Y \times Y) \\ \downarrow & & \downarrow \\ \mathrm{sub}(W \times \Phi Y) & \xrightarrow{\exists d} & \mathrm{sub}(W \times \Phi Y \times \Phi Y). \end{array}$$

Proof. The first commutes because a semilattice isomorphism preserves the largest element of the semilattice. For the second, observe that p is actually the value at Y of a natural transformation between the functor $X\times -$ and the constant functor X. Thus the diagram

$$\begin{array}{ccc} \mathrm{sub}(X\times Y) & \xleftarrow{\;p^*\;} & \mathrm{sub}(X) \\ \downarrow & & \downarrow \\ \mathrm{sub}(W\times \Phi Y) & \xleftarrow{\;p^*\;} & \mathrm{sub}(W) \end{array}$$

commutes. But the vertical arrows are isomorphisms and hence commute with the left adjoints $\exists p$ as well. The remaining two are similar.

This result, combined with the previous proposition, implies Theorem C.

Proposition 5. *For $E\in\mathscr{E}$ functors*

$$\mathrm{sub}(\Lambda E\times -),\ \mathrm{sub}(E\times\Delta\ -):\mathscr{B}^{\mathrm{op}}\to\mathscr{S}emilattice$$

are naturally equivalent.

Proof. We know that

$$\Omega^{\Lambda E}\cong\Gamma(\Omega^{E})$$

are isomorphic as semilattice objects of $\mathscr{B}$. This means that

$$\mathrm{Hom}(-,\Omega^{\Lambda E}),\ \mathrm{Hom}(-,\Gamma(\Omega^{E})):\mathscr{B}^{\mathrm{op}}\to\mathscr{S}emilattice$$

are naturally equivalent. But

$$\mathrm{Hom}(B,\Omega^{\Lambda E})\cong\mathrm{Hom}(\Lambda E\times B,\Omega)\cong\mathrm{sub}(\Lambda E\times B)$$

while

$$\mathrm{Hom}(B,\Gamma(\Omega^{E}))\cong\mathrm{Hom}(\Lambda B,\Omega^{E})\cong\mathrm{Hom}(E\times\Delta B,\Omega)\cong\mathrm{sub}(E\times\Delta B).$$

The value of $\mathrm{Hom}(\Lambda E\times -,\Omega)$ and $\mathrm{Hom}(E\times\Delta\ -,\Omega)$ on maps are computed by pulling back so that these functors are isomorphic in the way required to apply Theorem C. Thus we see that

$$\mathrm{Hom}(\Lambda E,B)\cong\mathrm{Hom}(E,\Lambda B)$$

which implies that Λ can be extended in a unique way to a functor left adjoint to Δ. We are ready to show that Δ is logical. Since we now know Λ is a functor all the above isomorphisms are natural in E. Then for all $E\in\mathscr{E}$,

$$\begin{aligned}\mathrm{Hom}(E,\Delta\Omega)&\cong\mathrm{Hom}(\Lambda E,\Omega)\cong\mathrm{sub}(\Lambda E)\\ &\cong\mathrm{sub}(E)\cong\mathrm{Hom}(E,\Omega)\end{aligned}$$

naturally in E so that $\Delta\Omega\cong\Omega$. Next, for $E\in\mathscr{E}$, $B\in\mathscr{B}$,

$$\mathrm{Hom}(E, \Delta(\Omega^B)) \cong (\Lambda E, \Omega^B) \cong \mathrm{Hom}(B, \Omega^{\Lambda E})$$

$$\cong \mathrm{Hom}(B, \Gamma(\Omega^E)) \cong \mathrm{Hom}(\Delta B, \Omega^E) \cong \mathrm{Hom}(E, \Delta\Omega^{\Delta B}),$$

so that $\Delta(\Omega^B) \cong \Delta\Omega^{\Delta B}$. Then for $E \in \mathscr{E}$, B', $B \in \mathscr{B}$, we have

$$\mathrm{Hom}(B', \Omega^{\Lambda(E\times\Delta B)}) \cong \mathrm{Hom}(\Lambda(E \times \Delta B), \Omega^{B'}) \cong \mathrm{Hom}(E \times \Delta B, \Delta\Omega^{\Delta B'})$$

$$\cong \mathrm{Hom}(E, \Delta\Omega^{\Delta(B\times B')}) \cong \mathrm{Hom}(E, \Delta(\Omega^{B\times B'}))$$

$$\cong \mathrm{Hom}(\Delta E, \Omega^{B\times B'}) \cong \mathrm{Hom}(B', \Omega^{\Lambda E\times B}).$$

Since Λ is determined uniquely by $\Omega^{\Lambda(\,)}$, it follows that

$$\Lambda(E \times \Delta B) \cong \Lambda E \times B.$$

Corollary. *For any* $B \in \mathscr{B}$, $\Lambda\Delta B \cong \Lambda 1 \times B$.

Thus the proof of Theorem B is completed by

Proposition 6. *The left adjoint* Δ *of a geometric morphism preserves exponentiation iff there is a natural equivalence*

$$\Lambda(E \times \Delta B) \cong \Lambda E \times B.$$

Proof. For any object B' of $\mathscr{B}$ we have the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}(B \times \Lambda E, B') & \cong & \mathrm{Hom}(\Lambda E, B'^{B}) & \cong & \mathrm{Hom}(E, \Delta(B'^{B})) \\
\downarrow & & & & \downarrow \\
\mathrm{Hom}(\Lambda(\Delta B \times E), B') & \cong & \mathrm{Hom}(\Delta B \times E, \Delta B') & \cong & \mathrm{Hom}(B, \Delta B'^{\Delta B}).
\end{array}$$

3. Further properties of Λ

In this section we derive further properties of a left adjoint to a logical functor.

Theorem D. *Let* Λ *be a left adjoint to a logical functor between toposes. Then* Λ *has the following properties*:

(i) Λ *preserves monomorphisms.*

(ii) Λ *creates* (*i.e. preserves and reflects*) *pullback of an arbitrary map with a monomorphisms.*

(iii) Λ *reflects epimorphisms.*

Proof. As remarked at the beginning of Section 2, a logical functor has a right adjoint iff it has a left adjoint. Thus we may suppose that $\Lambda \dashv \Delta \dashv \Gamma$ and, from Theorem B, that there is a natural equivalence

$$\Gamma(\Omega^E) \cong \Omega^{\Lambda E}.$$

To prove (i) we begin with,

Proposition 7. *A map $f: E \to F$ in a topos $\mathscr{E}$ is monomorphism (resp. epimorphism) iff*

$$\exists f: \Omega^E \to \Omega^F$$

is a split monomorphisms (resp. split epimorphism).

Proof. First off, note that $\exists f \cdot \Omega^f \cdot \exists f = \exists f$ always. Hence $\exists f$ is mono (resp. epi) iff $\Omega^f \cdot \exists f = \mathrm{Id}$ (resp. $\exists f \cdot \Omega^f = \mathrm{Id}$). Now if f is mono, subobjects of $E \times -$ are mapped monomorphically by image to subobjects of $F \times -$. If, conversely, f is not mono, the kernel pair of f and the diagonal of $E \times E$ are mapped to the same subobject of $E \times F$, namely the graph of f. If f is epi, every subobject of $F \times -$ is the image of one from $E \times -$ so that $\exists f$ is epi and conversely.

Now suppose that $f: E' \to E$ is a monomorphism. We have a commutative square

$$\begin{array}{ccc} \Gamma(\Omega^E) & \cong & \Omega^{\Lambda E} \\ {\scriptstyle \Gamma(\Omega^f)}\big\downarrow & & \big\downarrow{\scriptstyle \Omega^{\Lambda f}} \\ \Gamma(\Omega^{E'}) & \cong & \Omega^{\Gamma E'} \end{array}$$

of lattice, i.e. category, objects of $\mathscr{B}$. Since left adjoints are unique, it follows that

$$\begin{array}{ccc} \Gamma(\Omega^E) & \cong & \Omega^{\Lambda E} \\ {\scriptstyle \Gamma(\exists f)}\big\uparrow & & \big\uparrow{\scriptstyle \exists \Lambda f} \\ \Gamma(\Omega^{E'}) & \cong & \Omega^{\Lambda E'} \end{array}$$

commutes as well. Next observe that when f is mono, then $\exists f$ and $\Gamma(\exists f) = \exists(\Lambda f)$ are split monos. By applying the converse of Proposition 7, we see that Λf must be mono.

Proposition 8. *For any monomorphism $f: E' \rightarrowtail E$, the characteristic map of f is the transpose of the map $\ulcorner \chi_f \urcorner$ which is the composite*

$$1 \to \Omega^{E'} \xrightarrow{\exists f} \Omega^E$$

in which the first map is name of the characteristic map of the identity of E'.

Proof. Apply Hom(1, −) to get

$$1 \to \mathrm{sub}(E') \xrightarrow{\mathrm{Im}_f} \mathrm{sub}(E).$$

The first map chooses the identity subobject of E' which is mapped to the inclusion by Im_f. This defines the transpose of the characteristic map.

In the proof of Proposition 7, we saw that under the isomorphism of $\Omega^{\Lambda -}$ with $\Gamma(\Omega^{-})$, $\exists(\Lambda f)$ corresponds to $\Gamma(\exists f)$. Together with the preceding we conclude that if $f\colon E' \rightarrowtail E$ has characteristic map $\chi_f\colon E \to \Omega$, then the square

$$\begin{array}{ccc} \Lambda E' & \longrightarrow & \Lambda E \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \Omega \end{array}$$

is a pullback. The right hand map is the composite of $\Lambda E \to \Lambda\Omega \cong \Lambda\Delta\Omega$ with the adjunction map $\Lambda\Delta\Omega \to \Omega$. Consider now a commutative diagram

$$(*) \qquad \begin{array}{ccc} F' & \longrightarrow & E' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ F & \xrightarrow{\ f\ } & E \end{array}$$

of $\mathscr{E}$. We have a diagram

$$\begin{array}{ccccc} & \nearrow^{\Gamma(\ulcorner\chi_g\urcorner)} & \Gamma(\Omega^E) & \cong & \Omega^{\Lambda E} \\ 1 & & \downarrow{\scriptstyle \Gamma(\Omega^f)} & & \downarrow{\scriptstyle \Omega^{\Lambda f}} \\ & \searrow_{\Gamma(\ulcorner\chi_h\urcorner)} & \Gamma(\Omega^F) & \cong & \Omega^{\Lambda F} \end{array}$$

in which the square commutes while by the preceding remark the triangle commutes iff $(*)$ is a pullback. If it is a pullback, the same argument applied to the composite triangle implies that $\Lambda(*)$ is a pullback. Conversely if $\Lambda(*)$ is a pullback the triangle commutes and its transpose under adjointness

$$\begin{array}{ccc} & \nearrow^{\ulcorner\chi_g\urcorner} & \Omega^E \\ 1 & & \downarrow \\ & \searrow_{\ulcorner\chi_f\urcorner} & \Omega^F \end{array}$$

does as well. Thus $(*)$ is a pullback.

Finally, suppose $f\colon E' \to E$ is such that Λf is epi. Then by Proposition 7, $\exists\Lambda f$ is split epi. Thus so are $\Gamma\exists f$, $\mathrm{Hom}(1, \Gamma\exists f)$, $\mathrm{Hom}(\Delta 1, \exists f)$ and $\mathrm{Hom}(1, \exists f)$. This means that there is a subobject of E' taken by f epimorphically to the identity subobject of E. This is only possible if f is epi.

Corollary 1. *For every mono $m: E' \rightarrowtail E$, the square*

$$\begin{array}{ccc} E' & \xrightarrow{\;m\;} & E \\ \downarrow & & \downarrow \\ \Delta\Lambda E' & \xrightarrow{\;\Delta\Lambda m\;} & \Delta\Lambda E \end{array}$$

is a pullback (the vertical maps are adjunctions).

Proof. Apply Λ, use the corollary to Proposition 5 of Section 2 and the fact that Λ reflects such pullbacks.

Corollary 2. *If*

$$\begin{array}{ccc} E' & \xrightarrow{\;f\;} & E \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ \Delta B' & \xrightarrow{\;\Lambda a\;} & \Delta B \end{array}$$

is a pullback, then so is

$$\begin{array}{ccc} \Lambda E' & \xrightarrow{\;\Lambda f\;} & \Lambda E \\ {\scriptstyle \tilde g'}\downarrow & & \downarrow{\scriptstyle \tilde g} \\ B' & \xrightarrow{\;a\;} & B \end{array}$$

where $\tilde{g}'$ and $\tilde{g}$ correspond to g' and g resp. under adjunction.

Proof. The hypothesis is easily seen to be equivalent to supposing

$$\begin{array}{ccc} E' & \xrightarrow{\;gf\;} & \Delta B \\ {\scriptstyle (g',f)}\downarrow & & \downarrow{\scriptstyle \delta} \\ \Delta B' \times E & \xrightarrow{\;\Delta a \times g\;} & \Delta B \times \Delta B \end{array}$$

to be a pullback where δ is the diagonal map. It, and hence (g', f), is a mono so we have that

$$\begin{array}{ccc} \Lambda E' & \xrightarrow{\Lambda g \cdot \Lambda f} & B \times \Lambda 1 \\ {\scriptstyle (\bar{g}', \Lambda f)} \downarrow & & \downarrow {\scriptstyle \delta \times \Lambda 1} \\ B' \times \Lambda E & \xrightarrow{a \times (\bar{g}, \Lambda !)} & B \times B \times \Lambda 1 \end{array}$$

is a pullback. Here ! is the terminal map on E. An easy diagram chase shows that

$$\begin{array}{ccc} \Lambda E' & \longrightarrow & B \\ {\scriptstyle (\bar{g}', \Lambda f)} \downarrow & & \downarrow {\scriptstyle \delta} \\ B' \times \Lambda E & \xrightarrow{a \times \bar{g}} & B \times B \end{array}$$

is a pullback from which the desired result follows.

At this point we can give another characterization of logical morphisms.

Theorem E. *Let $\Delta : \mathscr{B} \to \mathscr{E}$ be a functor between toposes with a left adjoint Λ. Then Δ is logical iff*

(i) *the natural map $\Lambda(\Delta B \times E) \to B \times \Lambda E$ is an isomorphism,*
(ii) *Λ preserves monomorphisms,*
(iii) *Λ preserves and reflects pullbacks along monomorphisms.*

Proof. The "only if" part is from Theorem D. To go the other way, we have already observed – see Proposition 6 – that (i) implies the preservation of exponentiation. Thus we must show that the natural map

$$d : \Delta\Omega \to \Omega$$

is an isomorphism. We begin by defining a map $s : \Omega \to \Delta\Omega$ which will be the inverse. We actually define the transpose $\bar{s} : \Lambda\Omega \to \Omega$ as the classifying map of the mono $\Lambda(\text{true}) : \Lambda 1 \to \Lambda\Omega$. Consider the commutative diagram

$$\begin{array}{ccccccc} \Omega & \xrightarrow{\eta\Omega} & \Delta\Lambda\Omega & \xrightarrow{\Delta\bar{s}} & \Delta\Omega & \xrightarrow{d} & \Omega \\ {\scriptstyle \text{true}} \uparrow & & \uparrow {\scriptstyle \Delta\Lambda(\text{true})} & & \uparrow {\scriptstyle \Delta(\text{true})} & & \uparrow {\scriptstyle \text{true}} \\ 1 & \xrightarrow{\eta 1} & \Delta\Lambda 1 & \xrightarrow{\Delta !} & \Delta 1 & \xrightarrow{!} & 1 \end{array}$$

The maps labeled η are the adjunction maps and $\bar{s} = \Delta\bar{s} \cdot \eta\Omega$ by definition. The right hand square is a pullback by definition of d. The middle one is as well from the definition of $\bar{s}$ and the fact that Δ preserves pullbacks. The left hand one is seen to be by applying Λ to it to get

$$\begin{array}{ccc} \Lambda\Omega & \xrightarrow{(\eta\Omega\cdot\Lambda 1)} & \Lambda\Omega\times\Lambda 1 \\ {\scriptstyle \Lambda(\text{true})}\uparrow & & \uparrow{\scriptstyle \Lambda(\text{true})\times\Lambda 1} \\ \Lambda 1 & \xrightarrow{(\eta\Omega\cdot\Lambda 1)} & \Lambda 1\times\Lambda 1 \end{array}$$

and using the reflection property of (iii). Hence the whole rectangle is a pullback so that $d\cdot s:\Omega\to\Omega$ classifies "true" and is therefore the identity.

To go the other way, apply Λ to

$$\Delta\Omega \xrightarrow{d} \Omega \xrightarrow{\eta\Omega} \Delta\Lambda\Omega \xrightarrow{\Delta\bar{s}} \Delta\Omega$$

to get

$$\Lambda\Delta\Omega \xrightarrow{\Lambda d} \Lambda\Omega \xrightarrow{\Lambda\eta\Omega} \Lambda\Delta\Lambda\Omega \xrightarrow{\Lambda\bar{s}} \Lambda\Delta\Omega.$$

If we compose this with the adjunction $\varepsilon\Omega\cdot\Lambda\Delta\Omega\to\Omega$, we get

$$\varepsilon\Omega\cdot\Lambda\bar{s}\cdot\Lambda\eta\Omega\cdot\Lambda d = \bar{s}\cdot\varepsilon\Lambda\Omega\cdot\Lambda\eta\Omega\cdot\Lambda d = \bar{s}\cdot\Lambda d.$$

In the diagram

$$\begin{array}{ccccc} \Omega\times\Lambda 1 \cong \Lambda\Delta\Omega & \xrightarrow{\Lambda d} & \Lambda\Omega & \xrightarrow{\bar{s}} & \Omega \\ {\scriptstyle \text{true}\times\Lambda 1}\uparrow \quad \uparrow & & \uparrow & & \uparrow \\ \Lambda 1 \cong \Lambda\Delta 1 & \xrightarrow{\Lambda !} & \Lambda 1 & \xrightarrow{!} & 1 \end{array}$$

The right hand square is a pullback by the definition of $\bar{s}$ while the middle is seen to be a pullback by the definition of d and the preservation property of Λ. The composite is thus a pullback and so is

$$\begin{array}{ccc} \Omega\times\Lambda 1 & \xrightarrow{\text{proj}} & \Omega \\ {\scriptstyle \text{true}\times\Lambda 1}\uparrow & & \uparrow{\scriptstyle \text{true}} \\ \Lambda 1 & \longrightarrow & 1 \end{array}$$

By the uniqueness of the classifying map, we see that $\bar{s}\cdot\Lambda d$ must be the projection, i.e. the adjunction map which is, of course, the transpose of the identity $\Delta\Omega\to\Delta\Omega$. Hence $s\cdot d$ is the identity and the proof is complete.

4. Proof of Theorem A

We suppose that $\mathscr{E}$ is an atomic topos. Then let $\mathscr{A}$ be the full subcategory whose objects are the atoms – those non-empty objects whose only proper subobject is empty.

Proposition 9. *The category $\mathscr{A}$ has a small skeleton.*

Proof. Let $\mathscr{G}$ be a generating family in $\mathscr{E}$. Then there is a $G \in \mathscr{G}$ and a map – necessarily an epi – $G \to A$. Since G is also a disjoint sum of atoms, there is an $A' \rightarrowtail G$ for some $A' \in \mathscr{A}$. Then we have an epimorphism $A' \to A$ with A' one of the atoms in the subobject lattice of G. Thus every atom is a quotient of at least one of the atoms in a subobject lattice of a $G \in \mathscr{G}$. Since $\mathscr{G}$ is small and a topos is co-well-powered it follows there is only a set of such quotients. The full subcategory $\mathscr{A}$ consisting of atoms has as covers all maps $A' \to A$. These are epis so that given

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ A' & \to & A \end{array}$$

we can complete it to a commutative square by any atom of the non-empty $A' \times_A B$. Thus $\mathscr{A}$ is an atomic site. Since every morphism of $\mathscr{A}$ is an epimorphism *in* $\mathscr{E}$, the canonical topology in $\mathscr{A}$ is the same as that induced on $\mathscr{A}$ by the canonical topology of $\mathscr{E}$. Thus $\mathscr{E}$ is the category of sheaves and the assertion (iii)$\Rightarrow$(i) of Theorem A follows. We have seen how Theorem B implies that (ii)$\Rightarrow$(iii). To see that (i)$\Rightarrow$(ii), let $\mathscr{A}$ be an atomic site and $\mathscr{E} = \tilde{\mathscr{A}}$. We first observe that every constant functor is a sheaf. In fact, constant functors are always separated and since a cover is refined by any map in it, the sheaf condition is immediate. Thus if i is the inclusion of presheaves into sheaves, the diagram

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\;i\;} & (\mathscr{A}^{\mathrm{op}}, \mathscr{S}ets) \\ {}_{\Delta}\nwarrow & & \nearrow_{\Delta} \\ & \mathscr{S}ets & \end{array}$$

commutes. It follows that $\Lambda = \underrightarrow{\lim} \cdot i$ is left adjoint to Δ. Moreover

$$\Lambda(E \times \Delta n) = \Lambda E \times n$$

is automatic since Λ preserves sums and crossing with n of Δn is the n-fold copower. It follows from Proposition 6 that Δ preserves the exponential. To see that Δ preserves Ω we must show that $\Delta 2 = \Delta(1+1) = \Delta 1 + \Delta 1 = 1 + 1 = 2$ is the subobject classifier of $\mathscr{E}$. But for $A \in \mathscr{A}$, $\Omega(A)$ is the set of subsheaves of A. If $\emptyset \neq E \rightarrowtail A$ is such a subsheaf, there is an $A' \to E \to A$ which is a cover in $\mathscr{A}$ hence an epimorphism in $\mathscr{E}$.

Thus $E \rightarrowtail A$ is an isomorphism. Hence there are exactly two elements $\emptyset$ and A in $\Omega(A)$ so that $\Omega(A) = \Delta 2(A)$. Thus $\Omega = \Delta 2$. This completes Theorem A.

Notice that this last argument implies that the topology on $\mathscr{A}$ is that of double negation, since the negation of every non-empty subobject of an atom is empty.

5. Pullback of a local homeomorphism

In this section we will show:

Theorem F. *If*

$$\begin{array}{ccc} \mathscr{E}' & \xrightarrow{(g^*,\, g_*)} & \mathscr{E} \\ {\scriptstyle (\Delta',\Gamma')}\downarrow & & \downarrow{\scriptstyle (\Delta,\Gamma)} \\ \mathscr{B}' & \xrightarrow{(f^*,\, f_*)} & \mathscr{B} \end{array}$$

is a pullback with (f^*, f_*) *bounded and* (Δ, Γ) *a local homeomorphism, then* (Δ', Γ') *is also a local homeomorphism.*

As with the existence of pullbacks (see [4]) the argument is given by considering two special cases: when (f^*, f_*) is the morphism associated with an internal functor category and when it is the inclusion of sheaves for a topology.

From [4] we know that if $\mathbf{C}$ is an internal category object of $\mathscr{B}$, then $\Delta\mathbf{C}$ is again one in $\mathscr{E}$ and that

$$\begin{array}{ccc} \mathscr{E}^{\Delta \mathbf{C}^{\mathrm{op}}} & \longrightarrow & \mathscr{E} \\ \downarrow & & \downarrow \\ \mathscr{B}^{\mathbf{C}^{\mathrm{op}}} & \longrightarrow & \mathscr{B} \end{array}$$

is a pullback. Note that Corollary 2 of Proposition 8 can be stated in the convenient form that the canonical map

$$\Lambda(E \times_{\Delta B} \Delta B') \to \Lambda E \times_B B'$$

is an isomorphism.

Now let C_0 and C_1 be the object of objects and the object of morphisms, respectively, of $\mathbf{C}$. An internal functor $F: \Delta\mathbf{C}^{\mathrm{op}} \to \mathscr{E}$ is given by a pair (p, φ) where $p: E \to \Delta C_0$, $\varphi: E \times_{\Delta C_0} \Delta C_1 \to E$, satisfying appropriate identities. Then $\tilde{p}: \Lambda E \to C_0$ together with

$$\Lambda\varphi: \Lambda(E \times_{\Delta C_0} \Delta C_1) \cong \Lambda E \times_{C_0} C_1 \to \Lambda E$$

gives an internal functor $\mathbf{C}^{\mathrm{op}} \to \mathscr{B}$ that we call $\Lambda^{\mathbf{C}^{\mathrm{op}}}F$. If $(B \to C_0,\ B \times_{C_0} C_1 \to B)$ is an object of $\mathscr{B}^{\mathbf{C}^{\mathrm{op}}}$, $\Delta^{\mathbf{C}^{\mathrm{op}}}$ applied to it is

$$(\Delta B \to \Delta C_0,\ \Delta B \times_{\Delta C_0} \Delta C_1 \cong \Delta(B \times_{C_0} C_1) \to \Delta B).$$

A map to this from $(E \to \Delta C_0,\ E \times_{\Delta C_0} \Delta C_1 \to E)$ is a commutative diagram

$$\begin{array}{ccc}
E \times_{\Delta C_0} \Delta C_1 & \longrightarrow & \Delta B \times_{\Delta C_0} \Delta C_1 \\
\downarrow & & \downarrow \\
E & \longrightarrow & \Delta B \\
& \searrow \quad \swarrow & \\
& \Delta C_0 &
\end{array}$$

which corresponds under transposition to

$$\begin{array}{ccc}
\Lambda E \times_{C_0} C_1 & \longrightarrow & B \times_{C_0} C_1 \\
\downarrow & & \downarrow \\
\Lambda E & \longrightarrow & B \\
& \searrow \quad \swarrow & \\
& C_0 &
\end{array}$$

so that $\Lambda^{\mathbf{C}^{\mathrm{op}}}$ is left adjoint to $\Delta^{\mathbf{C}^{\mathrm{op}}}$. It is easily checked that for $F \in \mathscr{E}^{\mathbf{C}^{\mathrm{op}}}$, $G \in \mathscr{B}^{\mathbf{C}^{\mathrm{op}}}$, $\Lambda^{\mathbf{C}^{\mathrm{op}}}(F \times \Delta^{\mathbf{C}^{\mathrm{op}}} G) \cong \Lambda^{\mathbf{C}^{\mathrm{op}}} F \times G$ and hence by Proposition 6, $\Delta^{\mathbf{C}^{\mathrm{op}}}$ preserves the exponential. To show that it is logical, we can apply theorem E by observing that monomorphisms and pullback diagrams are "pointwise" notions. Since in $\mathscr{E}$ they are preserved (resp. created) by Λ, they are preserved (resp. created) by $\Lambda^{\mathbf{C}^{\mathrm{op}}}$ in $\mathscr{E}^{\mathbf{C}^{\mathrm{op}}}$. This shows the first case of Theorem F, that of functor categories.

Next we consider the case

$$\begin{array}{ccc}
& & \mathscr{E} \\
& & \downarrow {\scriptstyle (\Delta,\Gamma)} \\
\mathscr{B}_j & \underset{a}{\overset{i}{\rightleftarrows}} & \mathscr{B}
\end{array}$$

where $j: \Omega \to \Omega$ is a topology in $\mathscr{B}$, i the inclusion of j sheaves and a the associated sheaf functor. As usual, we let $J \rightarrowtail \Omega$ be the subject classified by j. Then J classifies j-dense monomorphisms. A topology is induced on the subobject classifier of $\mathscr{E}$ by the characteristic map of the image of

$$\Delta J \rightarrowtail \Delta\Omega \xrightarrow{d} \Omega.$$

In general this is not a topology but generates one. In this case d is an isomorphism and the induced topology is simply Δ_j. As shown in [3] the pullback we seek is the category of sheaves for this induced topology. In the present case, this means that

$$\begin{array}{ccc} \mathscr{E}_{\Delta j} & \overset{i}{\underset{a}{\rightleftarrows}} & \mathscr{E} \\ {\scriptstyle(\Delta_j,\Gamma_j)}\downarrow & & \downarrow{\scriptstyle(\Delta,\Gamma)} \\ \mathscr{B}_j & \overset{i}{\underset{a}{\rightleftarrows}} & \mathscr{B} \end{array}$$

is a pullback. Since the subobject classifier Ω_j in $\mathscr{B}_j$ is the equalizer of Id, $j: \Omega \to \Omega$ and Δ preserves Ω, it follows immediately that $\Delta_j \Omega_j = \Omega_{\Delta j}$. Next we claim $\Delta \cdot i = i \cdot \Delta_j$. To show this it is sufficient to show that when B is a j sheaf, ΔB is a Δ_j sheaf. But j sheaves are characterized by the property that when

$$\begin{array}{ccc} B'' & \rightarrowtail & B' \\ \downarrow & & \downarrow \\ 1 & \rightarrowtail & J \end{array}$$

is a pullback,

$$\mathrm{Hom}(B', B) \to \mathrm{Hom}(B'', B)$$

is an isomorphism. (See [6, 3.2] for details.) Now since

$$\begin{array}{ccc} J & \rightarrowtail & \Omega \\ \downarrow & & \downarrow{\scriptstyle j} \\ 1 & \rightarrowtail & \Omega \end{array}$$

is a pullback, so is

$$\begin{array}{ccc} \Delta J & \rightarrowtail & \Omega \\ \downarrow & & \downarrow{\scriptstyle \Delta j} \\ 1 & \rightarrowtail & \Omega \end{array}$$

Thus $E' \rightarrowtail E$ is Δ_j dense just in case there is a pullback

$$\begin{array}{ccc} E' & \rightarrowtail & E \\ \downarrow & & \downarrow \\ 1 & \rightarrowtail & \Delta J \end{array}$$

It follows from Corollary 2 of Proposition 8 that

$$\begin{array}{ccc} \Lambda E' & \rightarrowtail & \Lambda E \\ \downarrow & & \downarrow \\ 1 & \rightarrowtail & J \end{array}$$

is also a pullback whence, since B is a j sheaf,

$$\begin{array}{ccc} \mathrm{Hom}(\Lambda E, B) & \longrightarrow & \mathrm{Hom}(\Lambda E', B) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Hom}(E, \Delta B) & \longrightarrow & \mathrm{Hom}(E', \Delta B) \end{array}$$

are also isomorphisms. Thus ΔB is a Δ_j sheaf. From this we may show that $\Delta_j = a\Lambda i$ is left adjoint to Δ_j. In fact

$$\mathrm{Hom}(\Lambda_j E, B) \cong \mathrm{Hom}(a\Lambda iE, B) \cong \mathrm{Hom}(\Lambda iE, iB)$$
$$\cong \mathrm{Hom}(iE, \Delta iB) \cong \mathrm{Hom}(iE, i\Delta_j B) \cong \mathrm{Hom}(E, \Delta_j B).$$

Finally,

$$\Lambda_j(E \times \Delta_j B) \cong a\Lambda i(E \times \Delta_j B)$$
$$\cong a\Lambda(iE \times i\Delta_j B) \cong a\Lambda(iE \times \Delta iB)$$
$$\cong a(\Lambda iE \times iB) \cong a\Lambda iE \times aiB \cong \Lambda_j E \times B$$

and Proposition 6 implies that Δ_j preserves the exponential. Since we have already seen it preserves Ω_j, Theorem F follows.

Corollary. *Let $\mathscr{A}$ be an atomic site, $\mathscr{B}$ a Grothendieck topos and* $\mathrm{Sh}(\mathscr{A}, \mathscr{B})$ *the category of sheaves on $\mathscr{A}$ with values in $\mathscr{B}$. Then the global sections functor*

$$\Gamma: \mathrm{Sh}(\mathscr{A}, \mathscr{B}) \to \mathscr{B}$$

together with its left adjoint Δ constitutes a local homeomorphism.

Proof. As seen in [4], the square

$$\begin{array}{ccc} \mathrm{Sh}(\mathscr{A}, \mathscr{B}) & \longrightarrow & \mathrm{Sh}(\mathscr{A}, \mathscr{S}) \\ \downarrow & & \downarrow \\ \mathscr{B} & \longrightarrow & \mathscr{S} \end{array}$$

is a pullback. Thus the result follows from Theorems A and F.

6. Connected local homeomorphisms

Let $(\Delta, \Gamma): \mathscr{E} \to \mathscr{B}$ be a local homeomorphism. It is called *connected* if $\Lambda 1 = 1$ (where $\Lambda \dashv \Delta$).

Proposition 10. *The local homeomorphism* (Δ, Γ) *is connected if* Δ *is full and faithful.*

Proof. It follows from Theorem E that

$$\Lambda \Delta B \cong B \times \Lambda 1 \cong B$$

for all B if $\Lambda 1 = 1$.

One obvious example of a local homeomorphism is a localization (slice)

$$\mathscr{B}/B \to \mathscr{B}.$$

Here $\Delta B' = B' \times B \to B$, $\Lambda(B' \to B) = B'$ and $\Gamma(B' \to B) = \Pi_{B \to 1} B'$. (See [6, 1.4] for details.)

It follows from generalities on pullbacks (see [5, SGA 4, IV. 5.10] for the classical case) that to give a geometric morphism $\mathscr{E} \to B/B$ is equivalent to giving one $(\Delta, \Gamma): \mathscr{E} \to \mathscr{B}$ together with a map in $\mathscr{E}$ of $1 \to \Delta B$. In particular, if there is a left adjoint Λ, the adjunction $1 \to \Delta\Lambda 1$ provides a canonical factorization

$$\mathscr{E} \to \mathscr{B}/\Lambda 1 \to \mathscr{B}.$$

The second factor is logical; we claim that when the original morphism is a local homeomorphism, the first factor is a connected local homeomorphism. In fact, we have a pullback

$$\begin{array}{ccc} E/\Delta\Lambda 1 & \longrightarrow & \mathscr{B}/\Lambda 1 \\ \downarrow & & \downarrow \\ \mathscr{E} & \longrightarrow & \mathscr{B} \end{array}$$

and by Theorem F all maps are local homeomorphisms. The morphism $\mathscr{E} \to \mathscr{B}/\Lambda 1$ above is actually the composite

$$\mathscr{E} \cong \mathscr{E}/1 \to \mathscr{E}/\Delta\Lambda 1 \to \mathscr{B}/\Lambda 1$$

with the first the logical morphism between slices obtained from the adjunction $1 \to \Delta\Lambda 1$. Moreover, the left adjoint $\mathscr{E} \to \mathscr{B}/\Lambda 1$ takes E to $\Lambda !: \Lambda E \to \Lambda 1$. In particular 1 goes to $\Lambda 1 \to \Lambda 1$, i.e. the terminal object of $\mathscr{B}/\Lambda 1$. Thus we have established,

Theorem G. *Every local homeomorphism factors as a connected local homeomorphism followed by a localization.*

Proposition 11. *The local homeomorphism* (Δ, Γ) *is a localization iff for every* E *in* $\mathscr{E}$, $E \to \Delta\Lambda E$ *is a monomorphism.*

Proof. If this is a localization, $\mathscr{E} \to \mathscr{B}/\Lambda 1$ is an equivalence. Then $\Delta\Lambda(B \to \Lambda 1)$ is

$$\begin{array}{ccc} B & \longrightarrow & B \times \Lambda 1 \\ & \searrow \quad \swarrow & \\ & \Lambda 1 & \end{array}$$

which is evidently mono. To go the other way, decompose (Δ, Γ) as

$$\mathscr{E} \xrightarrow{(\Delta', \Gamma')} \mathscr{B}/\Lambda 1 \to \mathscr{B}.$$

We know that Δ' is full and faithful so that

$$\Lambda'\Delta'(B \to \Lambda 1) \to (B \to \Lambda 1)$$

is an isomorphism. Now apply Λ' to

$$E \rightarrowtail \Delta'\Lambda'E$$

to get an isomorphism

$$\Lambda'E \to \Lambda'\Delta'\Lambda'E \cong \Lambda'E.$$

By Theorem D, Λ' reflects epis so that $E \xrightarrow{\cong} \Delta'\Lambda'E$.

Corollary. *Suppose* $(\Delta, \Gamma): \mathscr{E} \to \mathscr{B}$ *is a local homeomorphism of toposes defined over* $\mathscr{S}$*ets and* $\mathscr{E}$ *is generated by subobjects of* 1. *Then* $\mathscr{E} \cong \mathscr{B}/\Lambda 1$.

Proof. For if $\eta: E \to \Delta\Lambda E$ is not mono, let $U \rightarrowtail 1$ and $f, g: U \to E$ two (mono-) morphisms such that $f \neq g$ but $\eta f = \eta g$. Since $\Lambda\eta$ is an isomorphism $\Lambda f = \Lambda g$. Now let

$$\begin{array}{ccc} V & \xrightarrow{h} & U \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{f} & X \end{array}$$

be a pullback. Apply Λ to get a pullback

$$\begin{array}{ccc} \Lambda V & \xrightarrow{\Lambda h} & \Lambda U \\ {\scriptstyle \Lambda k}\downarrow & & \downarrow{\scriptstyle \Lambda g} \\ \Lambda U & \xrightarrow{\Lambda f} & \Lambda X \end{array}$$

by Theorem D. Since $\Lambda g = \Lambda f$ is a mono, Λh and Λk are isomorphisms. Hence h and k are epimorphisms. Generalities on factorization systems imply the existence of an isomorphism $U \to U$ making both diagrams commute. But U being a subobject of 1, this map is the identity whence $f = g$.

7. Examples

In this section we give some examples of local homeomorphisms and atomic sites.

(1) *Maps between spatial toposes.* We show if $f: X \to Y$ is a continuous map between sober spaces, then

$$(f^*, f_*): \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$$

is a local homeomorphism iff f is a local homeomorphism. In fact, if f is a local homeomorphism, then $f: X \to Y$ is an object of $\mathrm{Sh}(Y)$ and quite evidently $\mathrm{Sh}(X) = \mathrm{Sh}(Y)/X$. Moreover the inverse image part of the local homeomorphism

$$\mathrm{Sh}(Y)/X \to \mathrm{Sh}(Y)$$

is gotten by pulling back along $X \to 1$ (i.e. $X \to Y$) which is precisely f^*. Since f^* determines its right and left adjoints, the result follows. Conversely, suppose f^* is logical. Then by the corollary to Proposition 11,

$$\mathrm{Sh}(X) = \mathrm{Sh}(Y)/W$$

for some $W \to Y$ in $\mathrm{Sh}(Y)$. In particular, the object $X \to X$ in $\mathrm{Sh}(X)$ is gotten by pulling back $X \to Y$ along f and the adjunction

$$f^* f_* 1 \to 1$$

is exactly $f:X \to Y$ which is thus a local homeomorphism. We have used that morphisms between Sh(X) and Sh(Y) are induced by continuous maps when X and Y are sober.

In particular when $Y=1$, we find that Sh(X) can be atomic only when X is discrete. In that case, of course $\mathrm{Sh}(X) = \mathcal{S}et/X$. Note that this implies that if B is a complete boolean algebra, then $\Gamma: \tilde{B} \to \mathcal{S}et$ has a logical left adjoint iff B is atomic.

Let B be a not-necessarily-complete boolean algebra. Since the boolean subalgebra generated by a finite number of elements is finite, B can be written as a filtered colimit $\varinjlim B_i$ where B_i is finite. If $B_i \to B_j$ is a transition map, the induced map between Stone spaces $\mathrm{St}(B_j) \to \mathrm{St}(B_i)$ is a local homeomorphism, each space being both finite and Hausdorff. Thus the induced $\tilde{B}_i \to \tilde{B}_j$ is a logical morphism. It follows that

$$\mathscr{E} = \varinjlim \tilde{B}_i$$

is a topos and it may easily be verified that $\mathscr{E}$ is a BVM for set theory, for which the global sections of 2 is the arbitrary boolean algebra B. (Traditional constructions of BVM's have required B to be complete.)

(2) Next we consider the question of characterizing categories $\mathscr{C}$ for which the functor category $\mathcal{S}et^{\mathscr{C}^{op}}$ is atomic. In fact we characterize categories $\mathscr{C}$ for which $\mathcal{S}et^{\mathscr{C}^{op}}$ is boolean. Let $\mathscr{C}$ be such a category and $f: A \to B$ be a morphism in $\mathscr{C}$. Let $R \rightarrowtail B$ be the subobject of the representable functor consisting of all maps which factor through f. Let R' be the complement of R and B. If the identity of B belongs to R', so does every map, which contradicts $f \in R$. Hence it belongs to R which is easily seen to imply that f is a split epi. Since f was arbitrary, every map in $\mathscr{C}$ – in particular the right inverse of f – is a split epi so that every map is an isomorphism. Thus $\mathscr{C}$ is a groupoid. The converse – that when $\mathscr{C}$ is a groupoid, $\mathcal{S}^{\mathscr{C}^{op}}$ is atomic – is left as an exercise.

(3) Given an arbitrary category $\mathscr{A}$, there is evidently at most one topology on $\mathscr{A}$ that makes it into an atomic site. There is such a topology iff the pullback of a non-empty sieve is non-empty, i.e. iff every pair of maps with common codomain can be completed to a commutative square. Given that $\mathscr{A}$ has such a topology, a more interesting question is whether it is subcanonical, that is whether the canonical functor $\mathscr{A} \to \tilde{A}$ is full and faithful. Of course, the answer is well known. Every sieve must be a universal epimorphic family (see, e.g. [1, appendix]). In this case, it means that *every* map must be a regular epimorphism (in the general sense of being the coequalizer of all pairs of maps that it coequalizes) along with the condition that every pair of maps with the same codomain be part of a commutative square. Such a category, equipped with the topology of non-empty sieves, will be called a standard atomic site. All the atomic sites we consider are standard.

(4) Here is an atomic site not associated with any groupoid action. Let X be an infinite set and M be the monoid of all surjective epimorphisms of X. Then M

considered as a category with the single object X is easily seen to be an atomic site. In fact as a category it even has pullbacks. In particular, every morphism is a regular epimorphism. [*Added in proof.* A. Joyal has observed that this *is* a category of G-sets.]

(5) For this and the following examples we use the fact (see [3]) that if $\mathscr{C}$ is a site there is a 1–1 correspondence between geometric morphisms $\mathscr{S}et \to \tilde{\mathscr{C}}$ and left exact functors $\mathscr{C} \to \mathscr{S}et$ which take covers to epimorphic families. Such a functor deserves to be called a *morphism of sites*. When $\mathscr{C}$ has finite limits, left exact means finite limit preserving. More generally, we call a set-valued functor left exact if it is a filtered colimit of representables. And if $\mathscr{C} \to \mathscr{D}$ is any functor, we say it is left exact if its composite with every left exact $\mathscr{D} \to \mathscr{S}et$ is left exact.

For the next example, we let $\mathscr{M}\mathscr{F}$ be the category whose objects are finite sets and maps are monomorphisms. $\mathscr{M}\mathscr{F}$ has the amalgamation property. In fact, if $l \leftarrowtail n \rightarrowtail k$ are two maps with common domain in $\mathscr{M}\mathscr{F}$, there is a commutative square

$$\begin{array}{ccc} n & \rightarrowtail & l \\ \downarrow & & \downarrow \\ k & \rightarrowtail & k \underset{n}{+} l \end{array}$$

in $\mathscr{M}\mathscr{F}$. (This is not the pushout in $\mathscr{M}\mathscr{F}$.) Thus $\mathscr{M}\mathscr{F}^{\mathrm{op}}$ is an atomic site. Let $\mathscr{E}$ be the category of sheaves. Then we claim that set-valued models on $\mathscr{E}$ are exactly finite sets. In fact, a geometric map $\mathscr{S}et \to \mathscr{E}$ is determined by a map of sites $\Phi : \mathscr{M}\mathscr{F}^{\mathrm{op}} \to \mathscr{S}et$. Suppose then we let $\Phi = \varinjlim \operatorname{Hom}(-, n_i)$. Then

$$\Phi(1) = \varinjlim \operatorname{Hom}(1, n_i) = \varinjlim n_i = N.$$

Moreover, for any k,

$$\Phi(k) = \varinjlim \operatorname{Hom}(k, n_i) \cong \operatorname{Mono}(k, \varinjlim n_i)$$

since k is finite and the limit is filtered. Of course $\varinjlim n_i$ will not necessarily (actually not every, see below) be finite but the formal identity is still valid. Thus $\Phi(k)$ is the set of monomorphisms of k to N. Moreover, for any $k \rightarrowtail l$ in $\mathscr{M}\mathscr{F}$, the corresponding

$$\Phi(l) \to \Phi(k)$$

must be a cover, i.e. an epimorphism in $\mathscr{S}et$. Thus the map

$$\operatorname{Mono}(l, N) \to \operatorname{Mono}(k, N)$$

must be surjective which is always possible iff N is infinite. Thus $\mathscr{E}$ classifies infinite sets.

(6) Let $\mathscr{M}\mathscr{F}\mathscr{O}_{0,0}$ be the category of finite total orders and strictly increasing maps. $\mathscr{M}\mathscr{F}\mathscr{O}_{1,0}$ is the category with the same objects and strictly increasing maps that

preserve the first element. Similarly for $i = 0, 1$, $\mathcal{MFO}_{i,1}$ is the subcategory of $\mathcal{MFO}_{i,0}$ of maps that preserve last element. Thus we have four categories $\mathcal{MFO}_{i,j}$ for $i = 0, 1$, $j = 0, 1$. In each case the dual category can be shown to be an atomic site. For example, the amalgamation property in $\mathcal{MFO}_{0,0}$ can be demonstrated as follows. Given

$$Z \xleftarrow{g} X \xrightarrow{f} Y,$$

define an order of $Y + Z$ by letting Y and Z have their given order while if $y \in Y$, $z \in Z$, we let $y < z$ unless $\exists x \in X$ with $z \leq g(x)$ and $f(x) \leq y$, in which case $z \leq y$. Let $\mathscr{E}_{i,j}$ be the topos of sheaves on $\mathcal{MFO}_{i,j}$. It is easily shown, using an argument similar to the above, that $\mathscr{E}_{0,0}$ classifies in $\mathscr{S}et$ dense total orders without first or last element. Similarly the other $\mathscr{E}_{i,j}$ classify dense total orders with first but not last element; with last but not first element; with both, respectively.

(7) Let k be a field and $\mathscr{SEX}_k$ be the category of separable extensions of k and k-linear field homeomorphisms. If $L_1 \leftarrow K \rightarrow L_2$ are two such extensions, $L_1 \otimes_K L_2$ is a product of a finite number of separable extensions of k, any of which suffices for the amalgamation. Thus $\mathscr{SEX}_k$ is the dual to an atomic site. If $\mathscr{E}$ is the category of sheaves, geometric morphisms $\mathscr{S}et$ to $\mathscr{E}$ are determined by morphisms of sites $\Phi: \mathscr{SEX}_k^{\mathrm{op}} \rightarrow \mathscr{S}et$. As usual, such a functor is a filtered colimit of representables,

$$\Phi(K) = \varinjlim \operatorname{Hom}(K, K_i)$$

If $\bar{k} = \lim K_i$, we have

$$\Phi(K) = \operatorname{Hom}(K, \bar{k})$$

since a k-linear map from K is determined by a finite number of elements. The fact that each K_i is separably algebraic over k implies $\bar{k}$ is, while the condition that a cover be taken to a cover implies that $\bar{k}$ is separably closed. Thus in sets, $\mathscr{E}$ classifies a separable closure of k. Notice that any two points of $\mathscr{E}$ are isomorphic.

Acknowledgements

We would like to acknowledge valuable discussions with Robert Paré. We should mention that André Joyal knew a form of Theorem A before we discovered it.

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