NON-SYMMETRIC $*$-AUTONOMOUS CATEGORIES

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1. Introduction

In [Barr, 1979] (hereafter known as SCAT) the theory of $*$-autonomous categories is outlined. Basically such a category is a symmetric monoidal category equipped with a strong duality equivalence to its opposite. These categories provide rich models of Girard's linear logic ([Girard, 1986], see also [Seely, 1989]). Linear logic is used to model, for example, the logic of resource use. However, it is still symmetric which means, for example, that it cannot be used to model temporal dependencies. For this and other reasons, it is interesting and useful to consider what remains of the theory when the symmetry is dropped.

Non-symmetric linear logic was actually the earliest version since it goes back to the syntactic calculus of [Lambek, 1958], although without the duality. The same author has recently returned to the subject in [Lambek, 1993], although his models are almost all lattice models.

An appendix to SCAT, written by P.-H. Chu outlined a formal construction by which a $*$-autonomous category could be constructed from any symmetric monoidal closed category that includes the original category fully. At the time, this construction was considered purely formal, but it has turned out to be one of the most interesting parts of the monograph. See [Barr, to appear]. We give here a non-symmetric version of this Chu construction.

2. Definitions

A monoidal category $\mathscr{A}$ is a category equipped with a tensor product $- \otimes -: \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$ that is coherently associative in the usual sense and for which there is a unit object we denote $\top$ and unit isomorphisms $l: \top \otimes A \longrightarrow A$ and $r: A \otimes \top \longrightarrow A$ that are natural in A. We will not name, nor explicitly use, the associativity isomorphisms, but will simply suppose that the tensor product is associative. It is known that every coherently associative category is equivalent to one that is actually associative, so this assumption is harmless and avoids a large amount of notational obscurity.

Since we are not supposing the tensor to be commutative, even up to isomorphism, the supposition that $- \otimes A$ have an adjoint is independent of the supposition that $A \otimes -$ have one. In fact, there is a tensor product on the category of uniform spaces that has

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one of the two for all objects, but not the other ([Isbell, 1964], Chapter 3). Therefore we distinguish, in this case, between a left closed, a right closed and a biclosed monoidal category. In this paper, we are concerned only with the last notion.

A biclosed monoidal category consists of a monoidal category as above, equipped with two functors $- \multimap -\colon \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \longrightarrow \mathscr{A}$ and $- \circ\!- -\colon \mathscr{A} \times \mathscr{A}^{\mathrm{op}} \longrightarrow \mathscr{A}$ for which there are natural isomorphisms $\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, C \circ\!- B) \cong \mathrm{Hom}(B, A \multimap C)$. These isomorphisms are mediated by "evaluation maps" $\mathrm{le}(A,B)\colon A \otimes (A \multimap B) \longrightarrow B$ and $\mathrm{re}(A,B)\colon (B \circ\!- A) \otimes A \longrightarrow B$, such that for any map $A \otimes B \longrightarrow C$, there are unique maps $A \longrightarrow C \circ\!- B$ and $B \longrightarrow A \multimap C$ such that the composites

$$A \otimes B \longrightarrow (C \circ\!- B) \otimes B \longrightarrow C$$

$$A \otimes B \longrightarrow A \otimes (A \multimap C) \longrightarrow C$$

are the given map. The two maps $A \otimes B \longrightarrow C$ and $A \longrightarrow C \circ\!- B$ are called transposes of each other as are $A \otimes B \longrightarrow C$ and $B \longrightarrow A \multimap C$. This term is thus ambiguous, but context will always clarify the situation.

A useful mnemonic over which way these isomorphisms go (as well as being the motivation for our choice, which is otherwise arbitrary) is to think of the simple example in which the category has as objects the elements of a group, with no non-identity arrows, $x \otimes y = xy$, $x \multimap y = x^{-1}y$ and $y \circ\!- x = yx^{-1}$. Motivated partly by examples like this, Lambek has used the notation A/B for $A \circ\!- B$ and $B\backslash A$ for $B \multimap A$. We will stick to the notation that has become common in linear logic.

The transpose of the composite arrow:

$$A \otimes (A \multimap B) \otimes (B \multimap C) \xrightarrow{\mathrm{le}(A,B)\otimes \mathrm{id}} B \otimes (B \multimap C) \xrightarrow{\mathrm{le}(B,C)} C$$

gives a map we denote

$$\mathrm{lc}(A,B,C)\colon (A \multimap B) \otimes (B \multimap C) \longrightarrow A \multimap C$$

and we similarly have a map

$$\mathrm{rc}(A,B,C)\colon (C \circ\!- B) \otimes (B \circ\!- A) \longrightarrow C \circ\!- A$$

which are the left and right composition arrows.

One thing it is useful to observe in a biclosed monoidal category is that from

$$\mathrm{Hom}(A \otimes D \otimes B, C) \cong \mathrm{Hom}(D \otimes B, A \multimap C) \cong \mathrm{Hom}(D, (A \multimap C) \circ\!- B)$$

and

$$\mathrm{Hom}(A \otimes D \otimes B, C) \cong \mathrm{Hom}(A \otimes D, C \circ\!- B) \cong \mathrm{Hom}(D, A \multimap (C \circ\!- B))$$

there is a "biclosed associativity" $A \multimap (C \circ\!- B) \cong (A \multimap C) \circ\!- B)$ and we will usually treat them as equal, writing $A \multimap C \circ\!- B$.

Another useful property is that the hom/tensor adjunction is internal. In fact, from

$$\operatorname{Hom}(D,(A\otimes B)\multimap C))\cong \operatorname{Hom}(A\otimes B\otimes D,C)\cong \operatorname{Hom}(B\otimes D,A\multimap C)$$
$$\cong \operatorname{Hom}(D,B\multimap(A\multimap C))$$

we conclude that $(A\otimes B)\multimap C\cong B\multimap(A\multimap C)$. Similarly, we have $C\multimapinv(A\otimes B)\cong (C\multimapinv B)\multimapinv A$.

There are at least four definitions of $*$-autonomous category and it will be useful to know that they are equivalent. We will give here all four and show later that they are equivalent. The first one is easily seen to be the non-symmetric form of the original definition given in SCAT, p. 13.

2.1. Definition A. A $*$-autonomous category is a biclosed monoidal category $\mathscr{A}$ together with a closed functor $(-)^*\colon\mathscr{A}\longrightarrow\mathscr{A}^{\mathrm{op}}$, which is a strong equivalence of categories.

The meaning of closed in this context is that $(-)^*$ is a functor between categories enriched over $\mathscr{A}$. $\mathscr{A}$ is enriched over itself using the closed structure of course. In the symmetric case, $\mathscr{A}^{\mathrm{op}}$ is enriched over $\mathscr{A}$ if you define $(A,B)=B\multimap A=A\multimapinv B$. In the non-symmetric case, these are distinct and only the second one works. Thus we define $(A,B)=A\multimapinv B$. The composition map $(A,B)\otimes(B,C)\longrightarrow(A,C)$ is just the composition map in $\mathscr{A}$, $(A\multimapinv B)\otimes(B\multimapinv C)\longrightarrow A\multimapinv C$. Then to say that $(-)^*$ is a strong isomorphism is to say that there is an isomorphism $\mathrm{t}(A,B)\colon A\multimap B\longrightarrow(A^*,B^*)=A^*\multimapinv B^*$ such that for all objects A, B and C, the diagram

$$\begin{tikzcd}
(A\multimap B)\otimes(B\multimap C) \arrow[r, "\mathrm{lc}(A,B,C)"] \arrow[d, "\mathrm{t}(A,C)\otimes\mathrm{t}(B,C)"'] & A\multimap C \arrow[d, "t(A,B)"] \\
(A^*\multimapinv B^*)\otimes(B^*\multimapinv C^*) \arrow[r, "\mathrm{rc}(C^*,B^*,A^*)"'] & A^*\multimapinv C^*
\end{tikzcd} \qquad (*)$$

commutes.

We have supposed that $(-)^*$ is an isomorphism and so it has an inverse. We denote it by ${}^*(-)$. All of its properties follow from properties of $(-)^*$. For instance, $B\multimapinv A\cong {}^*B\multimap{}^*A$ and a diagram analogous to $(*)$ commutes.

2.2. Definition B. Let $\mathscr{A}$ be a biclosed monoidal category. An object $\perp$ is called a dualizing object if for any object A the natural map $A\longrightarrow\perp\multimapinv(A\multimap\perp)$ gotten by transposing twice $\mathrm{id}\colon A\multimap\perp\longrightarrow A\multimap\perp$ is an isomorphism. Then a $*$-autonomous category is a biclosed monoidal category together with a dualizing object.

2.3. Definition C. A $*$-autonomous category is a monoidal category $\mathscr{A}$ equipped with an equivalence $(-)^*\colon\mathscr{A}\longrightarrow\mathscr{A}^{\mathrm{op}}$ such that there is a natural isomorphism

$$\operatorname{Hom}(A,B^*)\longrightarrow\operatorname{Hom}(\top,(A\otimes B)^*)$$

In this formulation, we do not even suppose that $\mathscr{A}$ is biclosed, rather deriving the internal homs from the tensor and star.

2.4. Definition D. A $*$-autonomous category is a closed category $\mathscr{A}$ in the sense of Eilenberg and Kelly [1966] together with an equivalence $(-)^*\colon \mathscr{A}^{\mathrm{op}} \longrightarrow \mathscr{A}$ that satisfies the condition

$$A \multimap (B \multimapinv C) \cong (A \multimap B) \multimapinv C$$

where $\multimapinv$ is the derived operator defined by $A \multimapinv B = A^* \multimap B^*$. This isomorphism should be an equivalence of functors of categories enriched over $\mathscr{A}$.

We note that the axioms for a closed category can be considerably simplified in the case that $\top$ is a generator, for then the underlying functor it represents is faithful. All that is needed are $\multimap$, $\top$, an isomorphism $u = uA\colon \top \multimap A \longrightarrow A$, natural in A and a composite $c = c(A,B,C)\colon B \multimap C \longrightarrow (A \multimap B) \multimap (A \multimap C)$ natural in all three arguments and subject to

1.
$$\mathrm{Hom}(A,B) \cong \mathrm{Hom}(\top, A \multimap B)$$

2.
$$\begin{tikzcd}
\mathrm{Hom}(\top, B \multimap C) \arrow[rr, "\mathrm{Hom}(\top,c)"] \arrow[d, "\cong"'] & & \mathrm{Hom}(\top, (A \multimap B) \multimap (A \multimap C)) \arrow[d, "\cong"] \\
\mathrm{Hom}(B,C) \arrow[rr] \arrow[dr] & & \mathrm{Hom}(A \multimap B, A \multimap C) \arrow[dl] \\
& \mathrm{Hom}(\mathrm{Hom}(A,B), \mathrm{Hom}(A,C)) &
\end{tikzcd}$$

The arrow in lower right is what results from applying the functor $\mathrm{Hom}(\top, -)$ to a map $A \multimap B \longrightarrow A \multimap C$.

3. The Chu construction

One of the most interesting constructions involving $*$-autonomous categories was that given by Chu that showed, among other things, that every monoidal closed category could be fully embedded into a $*$-autonomous category. Here we give an analagous construction for biclosed monoidal categories.

The Chu construction invariably requires the existence of certain pullbacks in the category we begin with and we assume the existence of these pullbacks without further mention. Practically, this limits the construction to the case of a category that has pullbacks.

One way of thinking about the original construction is that it simply adjoins to the category all possible duals of an object. In the non-symmetric case, you need not only the object A^*, but also A^{**}, A^{***}, $\ldots$, not to mention ${}^*A$, $\ldots$. Thus the construction is a good deal more complicated and involves not pairs of objects, but a doubly infinite sequence.

Let $\mathscr{V}$ be a biclosed monoidal category and $\perp$ an object of $\mathscr{V}$. We define a category $\mathscr{A}$ whose objects are doubly infinite indexed sequences $\mathbf{V} = (\ldots, V_{-1}, V_0, V_1, \ldots)$ of objects equipped with maps $V_i \otimes V_{i+1} \longrightarrow \perp$ for all $i \in \mathbf{Z}$. We will usually write $\mathbf{V} = (V_i)$ or $\mathbf{V} = (\mathbf{V})_i$. If $\mathbf{W} = (W_i)$ is another object, a morphism $\mathbf{f} = (f_i) \colon \mathbf{V} \longrightarrow \mathbf{W}$ consists of maps $f_{2i} \colon V_{2i} \longrightarrow W_{2i}$ for all $i \in Z$ and maps $f_{2i+1} \colon W_{2i+1} \longrightarrow V_{2i+1}$ for all $i \in \mathbf{Z}$ such that for all i, the following diagrams commute.

$$\begin{tikzcd}
V_{2i} \otimes W_{2i+1} \arrow[r, "V_{2i}\otimes f_{2i+1}"] \arrow[d, "f_{2i}\otimes W_{2i+1}"'] & V_{2i} \otimes V_{2i+1} \arrow[d] \\
W_{2i} \otimes W_{2i+1} \arrow[r] & \perp
\end{tikzcd}
\qquad
\begin{tikzcd}
W_{2i-1} \otimes V_{2i} \arrow[r, "f_{2i-1}\otimes V_{2i}"] \arrow[d, "W_{2i-1}\otimes f_{2i}"'] & V_{2i-1} \otimes V_{2i} \arrow[d] \\
W_{2i-1} \otimes W_{2i} \arrow[r] & \perp
\end{tikzcd}$$

We define functors $(-)^* \colon \mathscr{A} \longrightarrow \mathscr{A}$ and ${}^*(-) \colon \mathscr{A} \longrightarrow \mathscr{A}$ by $(\mathbf{V}^*)_i = (\mathbf{V})_{i-1}$ and $({}^*\mathbf{V})_i = (\mathbf{V})_{i+1}$. It is clear that both $(-)^*$ and ${}^*(-)$ are equivalences between $\mathscr{A}$ and $\mathscr{A}^{\mathrm{op}}$ and are inverse to each other.

3.1. The tensor product. Let $\mathbf{V} = (V_i)$ and $\mathbf{W} = (W_i)$ be objects of $\mathscr{A}$. We define $\mathbf{V} \otimes \mathbf{W}$ to be the object with $(\mathbf{V} \otimes \mathbf{W})_{2i} = V_{2i} \otimes W_{2i}$ while $(\mathbf{V} \otimes \mathbf{W})_{2i+1}$ is defined as the pullback

$$\begin{tikzcd}
(\mathbf{V} \otimes \mathbf{W})_{2i+1} \arrow[r] \arrow[d] & W_{2i} \multimap V_{2i+1} \arrow[d] \\
W_{2i+1} \multimapinv V_{2i+2} \arrow[r] & W_{2i} \multimap \perp \multimapinv V_{2i+2}
\end{tikzcd}$$

the right hand map is gotten by applying $W_{2i} \multimap -$ to the map $V_{2i+1} \longrightarrow \perp \multimapinv V_{2i+2}$ gotten by transposing $V_{2i+1} \otimes V_{2i+2} \longrightarrow \perp$. The bottom map is similar. The structure maps are given by

$$V_{2i} \otimes W_{2i} \otimes (\mathbf{V} \otimes \mathbf{W})_{2i+1} \longrightarrow V_{2i} \otimes W_{2i} \otimes (W_{2i} \multimap V_{2i+1}) \longrightarrow V_{2i} \otimes V_{2i+1} \longrightarrow \perp$$

and

$$(\mathbf{V} \otimes \mathbf{W})_{2i-1} \otimes V_{2i} \otimes W_{2i} \longrightarrow (W_{2i-1} \multimap V_{2i}) \otimes V_{2i} \otimes W_{2i} \longrightarrow W_{2i-1} \otimes W_{2i} \longrightarrow \perp$$

using the two projections on the pullback.

3.2. Proposition. *The tensor product is, up to natural isomophism, associative.*

Proof. If $\mathbf{U} = (U_i)$ is another object of $\mathscr{A}$, then there is no problem with $(\mathbf{U}\otimes\mathbf{V}\otimes\mathbf{W})_{2i} = U_{2i} \otimes V_{2i} \otimes W_{2i}$. To compute $(\mathbf{U} \otimes (\mathbf{V} \otimes \mathbf{W}))_{2i+1}$ we replace $(\mathbf{V} \otimes \mathbf{W})_{2i+1}$ by definition and use the fact that the functor $- \multimapinv U_{2i+2}$ preserves limits. Then the diagram

$$\begin{tikzcd}
 & & (\mathbf{U} \otimes (\mathbf{V} \otimes \mathbf{W}))_{2i+1} \arrow[dll] \arrow[d] \arrow[drr] & & \\
(W_{2i+1} \multimapinv V_{2i+2}) \multimapinv U_{2i+2} \arrow[dr] & & (W_{2i} \multimap V_{2i+1}) \multimapinv U_{2i+2} \arrow[dl] \arrow[dr] & & (V_{2i} \otimes W_{2i}) \multimap U_{2i+1} \arrow[dl] \\
 & (W_{2i} \multimap \perp \multimapinv V_{2i+2}) \multimapinv U_{2i+2} & & (V_{2i} \otimes W_{2i}) \multimap \perp \multimapinv U_{2i+2} &
\end{tikzcd}$$

is a limit. In a similar way we use the fact that $- \multimapinv W_{2i}$ preserves limits to conclude that

$$\begin{tikzcd}
 & & ((\mathbf{U} \otimes \mathbf{V}) \otimes \mathbf{W})_{2i+1} \arrow[dll] \arrow[d] \arrow[drr] & & \\
W_{2i+1} \multimapinv (U_{2i+2} \otimes V_{2i+2}) \arrow[dr] & & W_{2i} \multimap (V_{2i+1} \multimapinv U_{2i+2}) \arrow[dl] \arrow[dr] & & W_{2i} \multimap (V_{2i} \multimap U_{2i+1}) \arrow[dl] \\
 & W_{2i} \multimap \perp \multimapinv (V_{2i+2} \multimapinv U_{2i+2}) & & W_{2i} \multimap (V_{2i} \multimap \perp \multimapinv U_{2i+2}) &
\end{tikzcd}$$

That these two are naturally equivalent follows from canonical ismorphisms of biclosed monoidal categories. ∎

3.3. Proposition. *The object* $\mathbf{T}$ *defined by* $(\mathbf{T})_{2i} = \top$ *and* $(\mathbf{T})_{2i+1} = \perp$ *for all* $i \in \mathbf{Z}$ *is a left and right unit for the tensor product.*

Proof. The even terms are clear. For the odd terms we see that

$$\begin{tikzcd}
(\mathbf{T} \otimes \mathbf{V})_{2i+1} \arrow[r] \arrow[d] & V_{2i} \multimap \perp \arrow[d] \\
V_{2i+1} \multimapinv \top \arrow[r] & V_{2i} \multimap \perp \multimapinv \top
\end{tikzcd}$$

is a pullback and with the right hand vertical arrow an isomorphism, so is the left hand one. A similar argument works on the other side. ∎

3.4. PROPOSITION. *There is a natural one-one correspondence between maps* $\mathbf{T} \longrightarrow (\mathbf{V} \otimes \mathbf{W})^*$ *and maps* $\mathbf{V} \longrightarrow \mathbf{W}^*$.

PROOF. We let $\mathbf{U} = (\mathbf{V} \otimes \mathbf{W})^*$. Then $\mathbf{U} = (U_i)$ where $U_{2i+1} = V_{2i} \otimes W_{2i}$ and

$$
\begin{tikzcd}
U_{2i} \arrow[r] \arrow[d] & V_{2i} \multimap W_{2i-1} \arrow[d] \\
V_{2i-1} \multimapinv W_{2i-2} \arrow[r] & V_{2i} \multimap \perp \multimapinv W_{2i-2}
\end{tikzcd}
$$

is a pullback. A map $\mathbf{T} \longrightarrow \mathbf{U}$ consists of arrows $\top \longrightarrow U_{2i}$ and $U_{2i+1} \longrightarrow \perp$ for all $i \in \mathbf{Z}$ subject to two commutativity conditions to which we return later. A map $\top \longrightarrow U_{2i}$ is a commutative square

$$
\begin{tikzcd}
\top \arrow[r] \arrow[d] & W_{2i-1} \multimap V_{2i} \arrow[d] \\
W_{2i-2} \multimapinv V_{2i-1} \arrow[r] & W_{2i-2} \multimap \perp \multimapinv V_{2i}
\end{tikzcd}
$$

which, given the fact that $\mathrm{Hom}(\top, U \multimap V) \cong \mathrm{Hom}(U, V) \cong \mathrm{Hom}(\top, V \multimapinv U)$ is equivalent to a pair of arrows $V_{2i} \longrightarrow W_{2i-1}$ and $W_{2i-2} \longrightarrow V_{2i-1}$ such that

$$
\begin{tikzcd}
W_{2i-2} \otimes V_{2i} \arrow[r] \arrow[d] & W_{2i-2} \otimes W_{2i-1} \arrow[d] \\
V_{2i-1} \otimes V_{2i} \arrow[r] & \perp
\end{tikzcd}
$$

commutes. These are the exact data and one of the two commutative diagrams required for a morphism $\mathbf{V} \longrightarrow \mathbf{W}^*$. The only missing fact is the commutation of

$$
\begin{tikzcd}
V_{2i} \otimes W_{2i} \arrow[r] \arrow[d] & V_{2i} \otimes V_{2i+1} \arrow[d] \\
W_{2i-1} \otimes W_{2i} \arrow[r] & \perp
\end{tikzcd}
$$

The diagrams left to later were the commutation of

$$
\begin{tikzcd}
(\top)_{2i} \otimes U_{2i+1} \arrow[r] \arrow[d] & U_{2i} \otimes U_{2i+1} \arrow[d] \\
(\top)_{2i} \otimes (\top)_{2i+1} \arrow[r] & \perp
\end{tikzcd}
\qquad
\begin{tikzcd}
U_{2i-1} \otimes (\top)_{2i} \arrow[r] \arrow[d] & U_{2i-1} \otimes U_{2i} \arrow[d] \\
(\top)_{2i-1} \otimes (\top)_{2i} \arrow[r] & \perp
\end{tikzcd}
$$

The left diagram becomes

$$\begin{tikzcd}
\top \otimes V_{2i} \otimes W_{2i} \arrow[r] \arrow[ddd] & U_{2i} \otimes V_{2i} \otimes W_{2i} \arrow[d] \\
 & (V_{2i} \circ\!\!- W_{2i-1}) \otimes V_{2i} \otimes W_{2i} \arrow[d] \\
 & W_{2i-1} \otimes W_{2i} \arrow[d] \\
\perp \arrow[r] & \perp
\end{tikzcd}$$

which means that

$$\begin{tikzcd}
V_{2i} \otimes W_{2i} \arrow[r] \arrow[dr] & W_{2i-1} \otimes W_{2i} \arrow[d] \\
 & \perp
\end{tikzcd}$$

commutes where the diagonal arrow comes from the given arrow $\mathsf{T} \longrightarrow \mathsf{U}$. Similarly, the commutativity of the right hand diagram is equivalent to that of

$$\begin{tikzcd}
V_{2i-2} \otimes W_{2i-2} \arrow[r] \arrow[dr] & V_{2i-2} \otimes V_{2i-1} \arrow[d] \\
 & \perp
\end{tikzcd}$$

By taking the case $2i$ of this second diagram we conclude that both triangles and hence the outer square of

$$\begin{tikzcd}
V_{2i} \otimes W_{2i} \arrow[r] \arrow[d] \arrow[dr] & V_{2i} \otimes V_{2i+1} \arrow[d] \\
W_{2i-1} \otimes W_{2i} \arrow[r] & \perp
\end{tikzcd}$$

commute. But this is exactly the data required to have a map $\mathsf{V} \longrightarrow \mathsf{W}^*$ and this completes the proof. ∎

We have now shown that Definition B is satisfied, so that $\mathscr{A}$ is a $*$-autonomous category.

4. The Chu construction with an cyclic dualizing object

Let $\mathscr{V}$ be a biclosed monoidal category. Let us say that an object $\perp$ of $\mathscr{V}$ is *cyclic* if the functors $- \multimap \perp$ and $\perp \multimapinv -$ are naturally equivalent. Under this hypothesis, the original construction described in [Chu, 1979] can be carried out with relatively little modification.

As in Chu's paper, we define a category $\mathscr{C}$ whose objects are pairs (V, V'), equipped with an arrow $V \otimes V' \longrightarrow \perp$. A morphism $(f, f'): (V, V') \longrightarrow (W, W')$ consists of $f: V \longrightarrow W$ and $f': W' \longrightarrow V'$ such that the square

$$\begin{tikzcd}
V \otimes W' \arrow[r, "V \otimes f'"] \arrow[d, "f \otimes W'"'] & V \otimes V' \arrow[d] \\
W \otimes W' \arrow[r, ";‘‘‘"'] & \perp
\end{tikzcd}$$

commutes.

Note that $\mathrm{Hom}(V \otimes V', \perp) \cong \mathrm{Hom}(V, V' \multimap \perp) \cong \mathrm{Hom}(V, \perp \multimapinv V') \cong \mathrm{Hom}(V' \otimes V, \perp)$ so that if (V, V') is an object of $\mathscr{A}$, so is $(V, V')^* = (V', V)$. The following proposition is immediate.

4.1. PROPOSITION. *For any objects* (V, V') *and* (W, W') *of* $\mathscr{A}$*, there is an isomorphism, natural in both* $\mathrm{Hom}((V, V'), (W, W')) \cong \mathrm{Hom}((W', W), (V', V))$ *so that* $()^*$ *is an isomorphism of* $\mathscr{A}$ *with* $\mathscr{A}^{\mathrm{op}}$. ■

There is a functor $\mathscr{A} \longrightarrow \mathscr{C}$ that takes the object (V, V') to the object $\mathbf{V} = (V_i)$ where $V_{2i} = V$ and $V_{2i+1} = V'$. The arrow $V_{2i} \otimes V_{2i+1} \longrightarrow \perp$ is the given one, while $V_{2i-1} \otimes V_{2i} \longrightarrow \perp$ uses the alternate form $V' \otimes V \longrightarrow \perp$. The proof of the following are straightforward.

4.2. THEOREM. *The inclusion* $\mathscr{A} \longrightarrow \mathscr{C}$ *is full and faithful.* ■

4.3. PROPOSITION. *The subcategory* $\mathscr{C} \subseteq \mathscr{A}$ *is closed under tensor, internal homs and star.* ■

The tensor product of two objects of $\mathscr{C}$ can be described explicitly. If $(U, U') = (V, V') \otimes (W, W')$, then $U = V \otimes W$ while

$$\begin{tikzcd}
U' \arrow[r] \arrow[d] & W' \multimapinv V \arrow[d] \\
W \multimap V' \arrow[r] & W \multimap \perp \multimapinv V
\end{tikzcd}$$

is a pullback.

5. Examples

5.1. The symmetric case. We can form the category $\mathscr{A}$ even in the symmetric case. We get a $*$-autonomous category in which $*$ is not involutive. The category $\mathscr{A}$ exists as well and is simply the original Chu construction.

5.2. The braided case. A monoidal category is said to be braided if there is a symmetry isomorphism $c(A,B)\colon A\otimes B \longrightarrow B\otimes A$ that satisfies all the usual rules except that $c(A,B)^{-1} \neq c(B,A)$, or at least not necessarily. Any dualizing object is still cyclic, so we can form the category $\mathscr{C}$. Thus the original Chu construction works just as well in that case.

5.3. Subsets of a group. A simple example is given by the set of subsets of a group. This is a monoidal category with $V\otimes W = VW = \{vw \| v\in V, w\in W\}$. The adjoints are given by $V \multimap W = \{u \| uV \subseteq W\}$ and $W \multimapinv V = \{u \| Vu \subseteq W\}$. It is not hard to see that a subset $\perp$ is cyclic if and only if it is invariant under conjugation. For example, the singleton subset consisting of the identity has this property. However, that is not very interesting since there will be hardly any (V,V') with $VV' \subseteq \perp$ in that case. If the group has a normal subgroup N of index 2, then we get the category whose objects are pairs (V,V') of subsets such that either both $V\subseteq H$ and $V'\subseteq H$ or $V\cap H = V'\cap H = \emptyset$.

Another possibility is to take the entire group as $\perp$. In this case, the category consists of all pairs of subsets (V,V'). The tensor product is given by $(V,V')\otimes(W,W') = (VW, (V\multimap W')\cap(V'\multimapinv W))$. Although only a poset, this gives an interesting example of a $*$-autonomous category.

5.4. Relations on a set. The poset of relations on a set with circle composition is a biclosed monoidal category. If R and S are two relations, then $R\multimap S = \{\langle x,y\rangle \| \langle y,z\rangle \in R \Rightarrow \langle x,z\rangle \in S\}$ and $S \multimapinv R = \{\langle x,y\rangle \| \langle z,x\rangle \in R \Rightarrow \langle z,y \in S\}$. There are exactly two choices for a cyclic dualizing object. The first is the total relation, which is terminal in the category. It is clear that the terminal object, if any, will always be cyclic. The other candidate is the inequality relation, $\neq$. First we will see that $\neq$ is cyclic. In fact, for any relation R, $R\multimap{\neq} = \{\langle x,y\rangle \| \langle y,z\rangle \in R \Rightarrow x\neq z\} = \{\langle x,y\rangle \in R \| \langle y,x\rangle \notin R\}$ while ${\neq}\multimapinv R = \{\langle x,y\rangle \in R \| \langle z,x\rangle \in R \Rightarrow z\neq y\}$ which is exactly the same thing.

Suppose that the relation T is cyclic. Then $R\circ S\subseteq T$ if and only if $S\circ R\subseteq T$ since the two sides are equivalent, respectively, to $R\subseteq S\multimap T$ and $R\subseteq T\multimapinv S$. Applied to singletons, we see that $\{\langle x,y\rangle\}\circ\{\langle y,z\rangle\}\subseteq T$ if and only if $\{\langle y,z\rangle\}\circ\{\langle x,y\rangle\}\subseteq T$. When $x{\neq}y$, the latter always holds and hence $\langle x,z\rangle\in T$ for all $x{\neq}z$. Thus $\neq\,\subseteq T$. This shows that T lies between $\neq$ and the total relation. Suppose now that for some x, $\langle x,x\rangle \in t$. For any y, $\{\langle y,x\rangle\}\circ\{\langle x,y\rangle\}\subseteq T$ if and only if $\{\langle x,y\rangle\}\circ\{\langle y,x\rangle\}\subseteq T$. Since the latter holds, so does the former and we conclude that $\langle y,y\rangle\in t$ for all y, so that T is the total relation.

If we take $T={\neq}$, then the category consists of pairs (R,R') such that $R\circ R'^{\mathrm{op}}=\emptyset$. The tensor product is given by $(R,R')\otimes(S,S') = (R\circ S, (R\multimap S')\cap(R'\multimapinv S))$.

5.5. Terminal objects. Let $\mathscr{V}$ be any biclosed monoidal category with a terminal object 1. The terminal object is certainly cyclic. If we take that for $\perp$, then the category $\mathscr{A}$ is the category of pairs (V, V') subject to no condition. The tensor product is given by $(V, V') \otimes (W, W') = (V \otimes W, (V \multimap W') \times (V' \circ\!\!- W))$.

5.6. Bimodules over a ring. This example is definitely not symmetric (except in very special cases). Take a ring R, which may even be commutative. An R-bimodule is an abelian group equipped with actions $R \otimes_{\mathbf{Z}} M \longrightarrow M$ and $M \otimes_{\mathbf{Z}} R \longrightarrow M$ such that $(rm)s = r(ms)$ for $r, s \in R$ and $m \in M$. A homomorphism of bimodules is required to preserve both left and right actions. This category is monoidal with $\otimes_R$ as tensor product. The convention that we have used here requires that $M \multimap N$ be the set of left R-linear functions from M to N. The right R-module structure on M induces a left R-module structure on the homset and the right R-module structure on N indues a right R-module structure on the homset. The map $M \otimes (M \multimap N)$ takes $m \otimes f \mapsto mf$, putting the map on the right. Thus the linearity comes out as $(rm)f = r(mf)$, making it look like an associative law. The other internal hom, $N \circ\!\!- M$ is the set of left R-linear homomorphisms and they come out to the left of their arguments and so satisfy $g(mr) = (gm)r$. This convention of putting left linear maps on the right and right linear maps on the left was adopted by Bass 30 years ago in his presentation of the Morita theorems.

Note that even when R is commutative, this tensor product is not symmetric. An example is given by the ring $R = \mathbf{Z}[x]$. An R-bimodule is an abelian group together with two commuting endomorphisms, one designated as the left and the other as the right action. Let M be the group $\mathbf{Z}$ with x acting as the identity on the left and as multiplication by 2 on the right. Let N have the reverse actions. Then as abelian groups, both $M \otimes_R N$ and $N \otimes_R M$ are both isomorphic to $\mathbf{Z}$. However the first has identity action on both sides and the second has multiplication by 2 on both sides.

6. Equivalence of the four definitions

In this section we show that the four definitions are equivalent. Suppose now we have a category $\mathscr{A}$ and functor $(-)^*\colon \mathscr{A} \longrightarrow \mathscr{A}^{\mathrm{op}}$ that satisfies the conditions of Definition A. We begin by noting that $\top$ is a two-sided unit and it therefore follows that both $\top \multimap -$ and $- \circ\!\!- \top$ are equivalent to the identity. Thus, if ${}^*(-)$ denotes the functor inverse to $(-)^*$,

$$ {}^*\top \cong \top \multimap {}^*\top \cong \top^* \circ\!\!- \top \cong \top^* $$

so that ${}^*\top \cong \top^*$ and we denote it by $\perp$.

Define $\mathrm{s}A\colon (A \multimap \perp) \longrightarrow A^*$ as the composite

$$ A \multimap \perp \xrightarrow{\mathrm{t}(A,\perp)} A^* \circ\!\!- \perp^* \xrightarrow{\mathrm{id} \circ\!- e\top} A^* \circ\!\!- \top \xrightarrow{vA^*} A^* $$

We specialize $(*)$ to the case $C = \perp$ to get the following commutative diagram in which the horizontal arrows are always given by composition.

$$
\begin{tikzcd}
(A \multimap B) \otimes (B \multimap \perp) \arrow[r] \arrow[d, "\mathrm{t}(A,B) \otimes \mathrm{t}(B,\perp)"'] & A \multimap \perp \arrow[d, "\mathrm{t}(A,\perp)"] \\
(A^* \multimap B^*) \otimes (B^* \multimap \perp^*) \arrow[r] \arrow[d, "\mathrm{id} \otimes (\mathrm{id} \multimap e(\top))"'] & A^* \multimap \perp^* \arrow[d, "\mathrm{id} \circ\!\!- e(\top)"] \\
(A^* \multimap B^*) \otimes (B^* \multimap \top) \arrow[r] \arrow[d, "vB^* \otimes \mathrm{id}"'] & A^* \multimap \top \arrow[d, "vA^*"] \\
(A^* \multimap B^*) \otimes B^* \arrow[r] & A^*
\end{tikzcd}
$$

which is to say that

$$
\begin{tikzcd}
(A \multimap B) \otimes (B \multimap \perp) \arrow[r] \arrow[d, "\mathrm{t}(A,B) \otimes \mathrm{s}B"'] & A \multimap \perp \arrow[d, "\mathrm{s}A"] \\
(A^* \multimap B^*) \otimes B^* \arrow[r] & A^*
\end{tikzcd}
$$

commutes.

Next we claim that the transposed diagram

$$
\begin{tikzcd}
(A \multimap B) \arrow[rr] \arrow[dr, "\mathrm{t}(A,B)"'] & & (A \multimap \perp) \circ\!\!- (B \multimap \perp) \arrow[dl, "\mathrm{s}A \circ\!\!- \mathrm{s}B^{-1}"] \\
& (A^* \circ\!\!- B^*) &
\end{tikzcd}
$$

also commutes. This is not quite routine. We begin with the fact that in any left closed monoidal category, for any arrows $f\colon X \longrightarrow X'$ and $g\colon Y \longrightarrow Y'$, the pentagon

$$
\begin{tikzcd}
X \arrow[r] \arrow[dd, "f"'] & (X \otimes Y) \circ\!\!- Y \arrow[dr, "(f \otimes g) \circ\!\!- \mathrm{id}"] & \\
& & (X' \otimes Y') \circ\!\!- Y \\
X' \arrow[r] & (X' \otimes Y') \circ\!\!- Y' \arrow[ur, "\mathrm{id} \circ\!\!- g"'] &
\end{tikzcd}
$$

commutes. In an interpretation, each takes an element $x \in X$ to $\lambda y \circ f(x) \otimes g(y)$. When g happens to be an isomorphism, this yields the commutativity of

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & (X \otimes Y) \multimapinv Y \arrow[d, "(f\otimes g)\multimapinv g^{-1}"] \\
X' \arrow[r] & (X' \otimes Y') \multimapinv Y'
\end{tikzcd}$$

In particular, when

$$\begin{tikzcd}
X \otimes Y \arrow[r] \arrow[d, "f\otimes g"'] & Z \arrow[d, "h"] \\
X' \otimes Y' \arrow[r] & Z
\end{tikzcd}$$

commutes and if g is an isomorphism, so does

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & (X \otimes Y) \multimapinv Y \arrow[r] \arrow[d, "(f\otimes g)\multimapinv g^{-1}" description] & Z \multimapinv Y \arrow[d, "h\multimapinv g^{-1}"] \\
X' \arrow[r] & (X' \otimes Y') \multimapinv Y' \arrow[r] & Z' \multimap Y'
\end{tikzcd}$$

Applied to our diagram above, this shows that

$$\begin{tikzcd}
A \multimap B \arrow[r] \arrow[d, "\mathrm{t}(A,B)"'] & (A \multimap \bot) \multimapinv (B \multimap \bot) \arrow[d, "\mathrm{s}A\multimapinv \mathrm{s}B^{-1}"] \\
A^* \multimapinv B^* \arrow[r] & A^* \multimapinv B^*
\end{tikzcd}$$

commutes. The bottom map is the transpose of application (degenerate case of composition) and is the identity by definition. If we let $A = \top$, we can conclude that

$$\begin{tikzcd}
\top \multimap B \arrow[rr] \arrow[dr] & & (\top \multimap \bot) \multimapinv (B \multimap \bot) \arrow[dl] \\
& \bot \multimapinv B^* &
\end{tikzcd}$$

commutes. The square

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & \bot \multimapinv (B \multimap \bot) \arrow[d] \\
\bot \multimapinv B^* \arrow[r] & (\bot \multimapinv \top) \multimapinv (B \multimap \bot)
\end{tikzcd}$$

commutes with the top arrow the canonical map, the bottom the transpose of composition and the vertical arrows induced by l. The result is that

$$\begin{tikzcd}
B \arrow[rr] \arrow[dr] & & \bot \multimapinv (B \multimap \bot) \arrow[dl] \\
& \bot \multimapinv B^* &
\end{tikzcd}$$

commutes. But both diagonal maps are isomorphisms and hence so is the horizontal arrow, which shows that Definition B is satisfied.

If $\mathscr{A}$ with dualizing object $\bot$ satisfies Definition B, then we define $A^* = A \multimap \bot$ and ${}^*A = \bot \multimapinv A$. Then clearly these are inverse equivalences. Moreover, we have

$$\begin{aligned}\mathrm{Hom}(\top, (B \otimes A)^*) &\cong \mathrm{Hom}(\top, (B \otimes A) \multimap \bot) \cong \mathrm{Hom}(B \otimes A \otimes \top, \bot) \\ &\cong \mathrm{Hom}(A, B \multimap \bot) = \mathrm{Hom}(A, B^*)\end{aligned}$$

and so Definition C is satisfied.

Now suppose that $\mathscr{A}$ is a $*$-autonomous category in the sense of Definition C. Since $(-)^*$ is an equivalence, we can suppose without loss of generality that it is actually an isomorphism and so has an inverse we denote ${}^*(-)$. This means that ${}^*(A^*) \cong ({}^*A)^* = A$.

It is known that any coherently monoidal category is equivalent to one that is actually associative and unitary. That is, for all objects A, B and C, $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ and the associativity isomorphism is the identity. Similarly, $\top \otimes A = A \otimes \top = A$ and the unitary isomorphisms are the identity. These assumptions are not, of course, necessary, but they will save us a lot of useless notation.

The first thing we note is that, for any object A,

$$\mathrm{Hom}(A, {}^*\top) \cong \mathrm{Hom}(\top, {}^*(\top \otimes A)) \cong \mathrm{Hom}(\top, {}^*A) \cong \mathrm{Hom}(A, \top^*)$$

the last isomorphism being the application of *. Since this is clearly natural in A, we conclude that $\top^* \cong {}^*\top$. We may and will assume that this is isomorphism is the identity and denote ${}^*\top = {}^*\top$ by $\bot$.

Since $*$ is an equivalence, an equivalent formulation of Definition C is that there is a natural equivalence, for all objects A and B, $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(\top, (A \otimes {}^*B)^*)$. We then have,

$$\mathrm{Hom}(A, B) \cong \mathrm{Hom}(B^*, A^*) \cong \mathrm{Hom}(B^* \otimes A, \bot) \cong \mathrm{Hom}(\top, {}^*(B^* \otimes A))$$

which shows that although Definition C was stated assymetrically, in terms of $()^*$ rather than ${}^*()$, it is, in fact, symmetric.

Using distributivity of tensor, we have

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(\top, (A \otimes B \otimes {}^*C)^* \cong \mathrm{Hom}(A, (B \otimes {}^*C)^*)$$

for all objects A, B and C of $\mathscr{A}$, so that if we define $C \circ\!- B = (B \otimes {}^*C)^*$, we see that $\mathscr{A}$ is left closed. We also have

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(C^*, (A \otimes B)^*) \cong \mathrm{Hom}(C^* \otimes A, B^*) \cong \mathrm{Hom}(B, {}^*(C^* \otimes A))$$

so that if we define $A \multimap C = {}^*(C^* \otimes A)$ we see that $\mathscr{A}$ is right closed as well. We see that

$$A \multimap \bot = {}^*(\bot^* \otimes A) = {}^*(\top \otimes A) = {}^*A$$

and similarly, $A^* = \bot \circ\!- A$.

For any objects A, B and C of $\mathscr{A}$, the evaluation map $\mathrm{lc}(A, B, C) \colon (A \multimap B) \otimes (B \multimap Z) \longrightarrow A \multimap Z$ transposes to a map we call

$$\phi(A, B, Z) \colon A \multimap B \longrightarrow (A \multimap Z) \circ\!- (B \multimap Z)$$

6.1. Lemma. *For any objects A, B and C of $\mathscr{A}$, the diagram*

$$\begin{tikzcd}
(A \multimap B) \otimes (B \multimap C) \arrow[r, "\mathrm{rc}(A,B,C)"] \arrow[d, "\phi(A,B,Z) \otimes \phi(B,C,Z)"'] & A \multimap C \arrow[d, "\phi(A,B,Z)"] \\
((A \multimap Z) \circ\!- (B \multimap Z)) \otimes ((B \multimap Z) \circ\!- (C \multimap Z)) \arrow[r, "\mathrm{lc}"'] & (A \multimap Z) \circ\!- (C \multimap Z)
\end{tikzcd} \qquad (*)$$

commutes.

Proof. We use the formal language for closed monoidal categories described by Jay ([1989]). In this language there are variables of each type and these are combined to make well formed formulas, also typed. For example, if a and f are variables of type A and $A \multimap B$, respectively then $af = \mathrm{le}(a, f)$ is an expression of type B. In a similar way, we can define right evaluation in these terms. If f and g are variables of type $A \multimap B$ and $B \multimap C$, respectively, then for a variable a of type A, we describe $a\mathrm{lc}(f, g) = a(f \circ g) = (af)g$. Similarly we can describe right composition. The map $\phi(A, B, Z)$ can be described by the formula $a((\phi(A, B, Z)f)g) = (af)g$. Here is how to interpret this formula. f is a variable of type $A \multimap B$, g is a variable of type $(B \multimap Z)$ and a is a variable of type A. $\phi(A, B, Z)f$ is an expression of type $(A \multimap Z) \circ\!- (B \multimap Z)$ which means it acts on the left of its argument to produce an argument of type $A \multimap Z)$. Thus $(\phi(A, B, Z)f)g$ is an expression of that type and acts on an expression of type A to produce one of type Z. The expression on the right hand side of that equation is also an expression of type Z and they are equal.

Let a, f, g and h be variables of type A, $A \multimap B$, $B \multimap C$ and $C \multimap Z$, respectively. Then going around the upper right path of the diagram, we get

$$a((\phi(A, C, Z)(f \circ g))h) = (a(f \circ g))h = ((af)g)h$$

The lower left path gives

$$\begin{aligned} a((\phi(A, B, Z)f \circ \phi(B, C, Z)g)h) &= a((\phi(A, B, Z)f)\phi(B, C, Z)g)h) \\ &= (af)((\phi(B, C, Z)g)h) = ((af)g)h \end{aligned}$$

so they are equal. ∎

Let $Z = \perp$ and $\mathrm{t}(A,B) = \phi(A,B,\perp)$. We conclude,

6.2. Corollary. *For any objects A, B and C of $\mathscr{A}$, the diagram*

$$\begin{tikzcd}
(A \multimap B) \otimes (B \multimap C) \arrow[r, "\mathrm{rc}(A,B,C)"] \arrow[d, "\mathrm{t}(A,B)\otimes\mathrm{t}(B,C)"'] & A \multimap C \arrow[d, "\mathrm{t}(A,B)"] \\
({}^*A \circ\!- {}^*B) \otimes ({}^*B \circ\!- {}^*C) \arrow[r, "\mathrm{lc}({}^*C,{}^*B,{}^*A)"'] & {}^*A \circ\!- {}^*C
\end{tikzcd}$$

One last point has to be made. We know that on objects, $A^* = \perp \circ\!- A$ and ${}^*A = A \multimap \perp$, but we also have to verify it on maps. In particular, we have to know that $\mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(B \multimap \perp, A \multimap \perp)$ and $\mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(\perp \circ\!- B, \perp \circ\!- A)$ are isomorphisms, given that the corresponding maps for $()^*$ and ${}^*()$ are. The reason this is necessary is that we have used the pointwise adjunctness theorem applied to the isomorphism $\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(B, {}^*(C^* \otimes A)$ to infer that on objects $A \multimap C = {}^*(C^* \otimes A)$. But the pointwise adjointness theorem does not use the functoriality of the right hand, rather providing its own definition on arrows. In fact, given $f\colon A' \longrightarrow A$, $f \multimap C\colon {}^*(C^* \otimes A) \longrightarrow {}^*(C^* \otimes A')$ is the unique arrow such that the diagram

$$\begin{tikzcd}
\mathrm{Hom}(A \otimes B, C) \arrow[r, "\mathrm{Hom}(f\otimes B,C)"] \arrow[d, "\cong"'] & \mathrm{Hom}(A' \otimes B, C) \arrow[d, "\cong"] \\
\mathrm{Hom}(B, {}^*(C^* \otimes A)) \arrow[r, "\mathrm{Hom}(B,f\multimap C)"'] & \mathrm{Hom}(B, {}^*(C^* \otimes A'))
\end{tikzcd}$$

commutes. That this unique arrow is ${}^*(C^* \otimes f)$ follows from the naturality of $()^*$ and ${}^*()$ applied in the diagram

$$\begin{tikzcd}
\mathrm{Hom}(A \otimes B, C) \arrow[r, "\mathrm{Hom}(f\otimes B,C)"] \arrow[d, "\cong"'] & \mathrm{Hom}(A' \otimes B, C) \arrow[d, "\cong"] \\
\mathrm{Hom}(B, {}^*(C^* \otimes A)) \arrow[r, "\mathrm{Hom}(B,f\multimap C)"'] & \mathrm{Hom}(B, {}^*(C^* \otimes A'))
\end{tikzcd}$$

Finally, we will show that Definitions C and D are equivalent. It is clear that C implies D, so suppose that $\mathscr{A}$ is $*$-autonomous in the sense of D. We denote the inverse of $(-)^*$ by ${}^*(-)$ and define $A \otimes B = (B \multimap {}^*A)^*$. The definition $A \circ\!- B = A^* \multimap B^*$ implies, by replacing A and B by ${}^*A$ and ${}^*B$, respectively, that $A \multimap B = {}^*A \circ\!- {}^*B$. Then

$$\begin{aligned}
(A \otimes B) \multimap C &\cong (B \multimap {}^*A)^* \multimap C \cong (B \multimap {}^*A) \circ\!- {}^*C \\
&\cong B \multimap ({}^*A \circ\!- {}^*C) \cong B \multimap (A \multimap C)
\end{aligned}$$

This shows not only that tensor is adjoint to Hom, but strongly so, which is well known to be equivalent to the associativity of $\otimes$. We have,

$$\mathrm{Hom}(A, {}^*B) \cong \mathrm{Hom}(\top, A \multimap {}^*B) = \mathrm{Hom}(\top, {}^*(B \otimes A))$$

So that Definition C is satisfied (with respect to ${}^*(-)$ instead of $(-)^*$).

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