ON DUALITY OF TOPOLOGICAL ABELIAN GROUPS

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Abstract. Let $\mathcal{G}$ denote the full subcategory of topological abelian groups consisting of the groups that can be embedded algebraically and topologically into a product of locally compact abelian groups. We show that there is a full coreflective subcategory $\mathcal{S}$ of $\mathcal{G}$ that contains all locally compact groups and is *-autonomous. This means that for all G, H in $\mathcal{S}$ there is an "internal hom" $G \multimap H$ whose underlying abelian group is $\mathrm{Hom}(G, H)$ and that that makes $\mathcal{S}$ into a closed category with a tensor product whose underlying abelian group is a quotient of the algebraic tensor product. Moreover a perfect duality results if we let $\mathbf{T}$ denote the circle group and define $G^* = G \multimap \mathbf{T}$. This is essentially a new exposition of work originally done jointly with H. Kleisli [Theory Appl. Categories, **8**, 54–62].

1. Introduction

We let $\mathcal{G}$ denote the category of those topological abelian groups that can be embedded, topologically and algebraically, into a product, possibly infinite, of locally compact abelian groups. The maps in $\mathcal{G}$ are the continuous homomorphisms. All groups considered in this paper will belong to $\mathcal{G}$ and, unless stated to the contrary, all homomorphisms will be continuous.

We let $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ denote the circle group. If $G \in \mathcal{G}$, we denote the underlying discrete group by $|G|$. We will write all groups additively. We will represent $\mathbf{T}$ as the interval $[-1, 1]$ with -1 and 1 identified. Addition is ordinary addition using the absolutely least residue mod 2. We let U be the open interval $(-1/2, 1/2)$ A character on G will be understood to be a homomorphism into $\mathbf{T}$. The (abstract) group of characters on G will be denoted $X(G)$.

The purpose of this note is to show that there is a full subcategory $\mathcal{S} \subseteq \mathcal{G}$ with the following properties:

1. $\mathcal{S}$ is complete and cocomplete.
2. If A and B belong to $\mathcal{S}$, there is a topology on $\mathrm{Hom}(A, B)$ that makes it into an object of $\mathcal{S}$. We denote this object by $A \multimap B$.
3. If A and B belong to $\mathcal{S}$, there is a topology on the tensor product $|A| \otimes |B|$ that makes it into an object of $\mathcal{S}$. We denote this object by $A \otimes B$.

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4. The usual isomorphism $|A| \otimes |B| \cong |B| \otimes |A|$ extends to an isomorphism $A \otimes B \cong B \otimes A$.

5. For any objects A, B, and C of $\mathcal{S}$, there is a canonical isomorphism $(A \otimes B) \multimap C \xrightarrow{\cong} A \multimap (B \multimap C)$.

6. For any object A of $\mathcal{S}$, the canonical map $A \longrightarrow (A \multimap \mathbf{T}) \multimap \mathbf{T}$ is an isomorphism.

Let us denote $A \multimap \mathbf{T}$ by A^*. Then point 6 above says simply that $A^{**} \cong A$. From points 5 and 6 above, we see that there is a canonical map $A \otimes A^* \multimap \mathbf{T}$. Applying the underlying set functor this produces the evaluation map $A \otimes A^* \longrightarrow \mathbf{T}$. Note that the continuity of this map does imply the continuity of $A \times A^* \longrightarrow \mathbf{T}$. That map is continuous in each argument, but presumably not jointly continous in general.

2. The main lemmas

The following facts are well known. We include them order to make this paper more accessible to category theorists.

2.1. Lemma. *The interval U has the following properties*

1. *U contains no non-zero subgroup.*

2. *For any object $G \in \mathcal{G}$, a homomorphism $\chi : |G| \longrightarrow \mathbf{T}$ is continuous on G if and only if $\chi^{-1}(U)$ is a neighbourhood of 0 in G.*

Proof. The first item is obvious. For the second, suppose that V is an open neighbourhood of 0 in G with $V \subseteq \chi^{-1}(U)$. Let U_n be the open interval $(-2^{-n}, 2^{-n})$, for $n > 0$ in $\mathbf{Z}$. The $\{U_n\}$ are a neighbourhood base of 0 in $\mathbf{T}$, so it is sufficient to show that $\chi^{-1}(U_n)$ is a neighbourhood of 0 in G for each n. By assumption, $\chi^{-1}(U_1)$ is a neighbourhood of 0 so suppose we have found an open neighbourhood V_n of 0 with $V_n \subseteq \varphi^{-1}(U_n)$. Choose an open neighbourhood V_{n+1} of 0 such that $V_{n+1} + V_{n+1} \subseteq U_n$. Then for $x \in V_{n+1}$, both $x = x + 0$ and $2x$ belong to V_n so that $\chi(x)$ and $2\chi(x)$ are in U_n. The latter fact implies that either $\chi(x)$ or $1 + \chi(x)$ belongs to U_{n+1} but the second possibility is precluded by the fact that $\chi(x) \in U_n$. ∎

Before continuing, we look at the paper [Moskowitz, 1967]. It defines a **proper morphism** of locally compact abelian groups as a continuous group homomorphism that is open onto its image. The inclusion of a closed subgroup, equipped with the subspace topology, is therefore proper. If H is a closed subgroup of G, the projection $p : G \longrightarrow G/H$ is also proper, which in the case of a surjection means open. For if $U \subseteq G$ is open, then $p^{-1}(p(U)) = \bigcup_{x \in H}(x + U)$ and translates of open sets are open. But then $p(U)$ is open in the quotient topology.

The paper then goes on to define injectivity as follows. The group I is injective if for every exact sequence of proper maps

$$0 \longrightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \longrightarrow 0$$

and every $g : G_1 \longrightarrow I$ there is a $\overline{g} : G_2 \longrightarrow I$ such that $\overline{g} f_1 = g$. Projectivity is defined dually. It is then shown (Theorem 3.1) that I is injective iff I^* is projective. Since $\mathbf{Z} = \mathbf{T}^*$ is certainly projective, it follows that $\mathbf{T}$ is injective. In fact, we can draw a somewhat better conclusion.

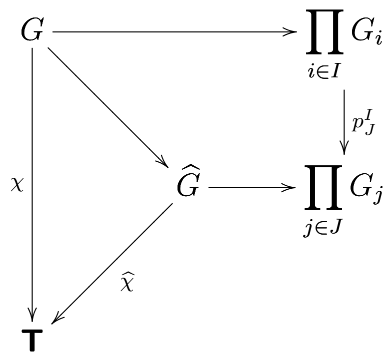
2.2. Theorem. *The group $\mathbf{T}$ is injective with respect to subgroup inclusions into a locally compact abelian group.*

Proof. If $A \subseteq G_2$ is an arbitrary subgroup and $h : A \longrightarrow \mathbf{T}$ is a continuous homomorphism, it is uniformly continuous in the standard group uniformities of A and $\mathbf{T}$. Since $\mathbf{T}$ is compact it is complete and therefore h can be extended uniquely to a continuous homomorphism $g : G_1 = \mathrm{cl}(A) \longrightarrow \mathbf{T}$. The extension to $\overline{g} : G_2 \longrightarrow \mathbf{T}$ then follows from the preceding discussion.

I would like to thank Milo Moses for calling my attention to the fact I had only asserted without proof the injectivity of $\mathbf{T}$ in the original version of this note.

In the statement of the following lemma, for $J \subseteq I$, the map $p^I_J : \prod_{i \in I} L_j \longrightarrow \prod_{j \in J} L_j$ is the projection of the product on the product over a subset of indices.

2.3. Lemma. *Suppose $G \subseteq \prod_{i \in I} G_i$ embeds G into a product of groups. For any character $\chi \in X(G)$, there is a finite subset $J \subseteq I$ such that for $\widehat{G}$ the image of the composite $G \longrightarrow \prod_{i \in I} G_i \xrightarrow{p^I_J} \prod_{j \in J}$ equipped with the subspace topology and a character $\widehat{\chi} \in X(\widehat{G})$ such that the diagram*



commutes.

Proof. Let $U \subseteq \mathbf{T}$ be the neighbourhood of 0 in $\mathbf{T}$ described above. Let $V = \chi^{-1}(U)$. Then V is a neighbourhood of 0 in G. Since G is a subgroup of $\prod G_i$, there is a neighbourhood of 0 in the product that meets G in V. There is thus a finite subset $J \subseteq I$ and open neighbourhoods $\{0 \in W_j \subseteq G_j \mid j \in J\}$ such that

$$G \cap \left(\prod_{j \in J} W_j \times \prod_{i \in I - J} G_i \right) \subseteq V$$

In particular, if we let

$$G_0 = G \cap \left(\prod_{j\in J} \{0\} \times \prod_{i \in I-J} G_i \right)$$

then G_0 is a subgroup of G and $\chi(G_0)$ is a subgroup of $\mathbf{T}$ contained in U and is hence 0. It is clear that $\widehat{G} = G/G_0$ embeds into $\prod_{j\in J} G_j$ and *we topologize it as a subgroup of* $\prod_{j\in J} G_j$. Let $\widehat{\chi} : \widehat{G} \longrightarrow \mathbf{T}$ be the induced homomorphism. It is continuous, even in the subspace topology, because

$$\widehat{\chi}^{-1}(U) \supseteq \widehat{G} \cap \prod_{j\in J} W_j \qquad \blacksquare$$

2.4. Corollary. *The group* $\mathbf{T}$ *is injective in* $\mathcal{G}$ *with respect to the class of topological inclusions.*

Proof. It is sufficient to show that $\mathbf{T}$ is injective with respect to the class of embeddings $G \subseteq \prod L_i$ in which each L_i is locally compact. Let $\chi \in X(G)$ and let J, $\widehat{G}$, and $\widehat{\chi}$ be as in the preceding lemma. Now let $\overline{G}$ be the toplogical closure of $\widehat{G}$ in $\prod_{j\in J} L_i$. Continuous homomorphisms of topological groups are uniformly continuous in the canonical (left) uniformity and $\mathbf{T}$ is compact, hence complete, so that $\widehat{\chi}$ extends to $\overline{\chi} : \overline{G} \longrightarrow \mathbf{T}$. But $\prod_{j\in J} L_i$ is a finite product of locally compact groups and is therefore locally compact, as is the closed subgroup $\overline{G}$. Injectivity of $\mathbf{T}$ in the category of locally compact groups implies that $\overline{\chi}$ extends to $\psi : \prod_{j\in J} L_j \longrightarrow \mathbf{T}$, as required. $\blacksquare$

2.5. Corollary. *Suppose* $G \subseteq \prod_{i\in I} G_i$ *embeds* G *into a product of abelian groups in* $\mathcal{G}$. *For any character* $\chi \in X(G)$, *there is a finite subset* $J \in I$ *and a character* $\psi : \prod_{j\in J} G_j \longrightarrow \mathbf{T}$ *such that the diagram*

$$\begin{array}{ccc} G & \hookrightarrow & \prod_{i\in I} G_i \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle p^I_J} \\ \mathbf{T} & \xleftarrow[\psi]{} & \prod_{j\in J} G_j \end{array}$$

commutes.

Proof. The only place that local compactness was used in the proof of 2.3 was for the injectivity of $\mathbf{T}$ in the category of locally compact groups. We have just seen that $\mathbf{T}$ is injective in $\mathcal{G}$, so the proof can be copied with the step from $\widehat{G}$ to $\overline{G}$ omitted. $\blacksquare$

2.6. Theorem. *Suppose* A *is an abelian group and* X *is a subgroup of* $\operatorname{Hom}(A, \mathbf{T})$ *that separates. Then there is at least one topology on* A *that makes into a group in* $\mathcal{G}$ *and whose group of continuous characters is* X. *Among all such topologies there is a coarsest and a finest.*

PROOF. Let $\sigma_X A$ denote the weak topology on A induced by X. By hypothesis, the map $f : A \longrightarrow \mathbf{T}^X$, defined by $p_\chi f = \chi$, is injective and $\sigma_X A$ embeds it as a topological subgroup. By 2.4, every character on $\sigma_X A$ extends to $\mathbf{T}^X$. So let χ be a character on $\mathbf{T}^X$. By 2.3, there is a finite subset $J \subseteq X$ such that χ factors through $\mathbf{T}^J$. Since the dual of $\mathbf{T}^J$ is $\mathbf{Z}^J$, it follows that there are integers $\{n_j \mid j \in J\}$ such that $\chi = \sum_{j\in J} n_j p_{\chi_j}$, and then $f\chi = \sum n_j \chi_j \in X$ since X is a subgroup of $\mathrm{Hom}(A, \mathbf{T})$.

For the finest, we let $\{G_i \mid i \in I\}$ range over all the topological groups in $\mathcal{G}$ that have the same pointset and abelian group structure as A and whose character group is X. In particular, for each $i \in I$, the identity map $G_i \longrightarrow \sigma_X A$ is continuous. Define $\tau_X A$ as the pullback in

$$\begin{array}{ccc} \tau_X A & \longrightarrow & \prod_{i\in I} G_i \\ \downarrow & & \downarrow (\mathrm{id}_A)^I \\ \sigma_X A & \xrightarrow[\Delta]{} & (\sigma_X A)^I \end{array}$$

Here Δ is the diagonal arrow. In other words, $\tau_X A$ is the subspace of $\prod G_i$ consisting of all the constant sequences $(x, x, x, \ldots)$ for $x \in A$, equipped with the subspace topology. Clearly the topology on $\tau_X A$ is the sup of those of the G_i. The only thing left to show is that $X(\tau_X A) = X$. If $\chi : \tau_X A \longrightarrow \mathbf{T}$ is a character, Lemma 2.5 says there is a finite subset $J \subseteq I$ and a character $\psi \in X\left(\prod_{j\in J} G_j\right)$ such that

$$\begin{array}{ccc} G & \hookrightarrow & \prod_{i\in I} G_i \\ \chi \downarrow & & \downarrow p \\ \mathbf{T} & \xleftarrow[\psi]{} & \prod_{j\in J} G_j \end{array}$$

commutes. But ψ is also a character on

$$\sigma\left(\prod_{j\in J} G_j\right) \cong \prod_{j\in J} \sigma G_j \cong (\sigma G)^J$$

(see 3.3) and its restriction to G is just χ. ■

2.7. NOTATION. To simplify notation, we will denote by σG, respectively τG, the groups $\sigma_{X(G)}|G|$, respectively $\tau_{X(G)}|G|$. Evidently, σG and τG are the coarsest and finest topologies on $|G|$ with the same characters as G.

2.8. THEOREM. *The object functions σ and τ extend to functors.*

Proof. For σ, this is obvious. For τ it is sufficient to show that given $G \longrightarrow H$ and and a weak isomorphism $H' \longrightarrow H$, there is a weak isomorphism $G' \longrightarrow G$ and a commutative diagram

$$\begin{array}{ccc} G' & \longrightarrow & G \\ \downarrow & & \downarrow \\ H' & \longrightarrow & H \end{array}$$

We let G' be a pullback $G \times_H H'$. It suffices to show that $H' \longrightarrow H$ is a weak isomorphism. Since H' is defined as a pullback, it is embedded in the product $G \times H'$. Then we have a commutative square

$$\begin{array}{ccc} G' & \hookrightarrow & G \times H' \\ \downarrow & & \downarrow \\ G & \longrightarrow & G \times H \end{array}$$

If we apply the functor $\mathrm{Hom}(-, R)$ we get the square

$$\begin{array}{ccc} \mathrm{Hom}(G', \mathbf{T}) & \twoheadleftarrow & \mathrm{Hom}(G, \mathbf{T}) \times \mathrm{Hom}(H', \mathbf{T}) \\ \uparrow & & \uparrow \cong \\ \mathrm{Hom}(G, \mathbf{T}) & \longleftarrow & \mathrm{Hom}(G, \mathbf{T}) \times \mathrm{Hom}(H, \mathbf{T}) \end{array}$$

from which it follows that $\mathrm{Hom}(G, \mathbf{T}) \longrightarrow \mathrm{Hom}(G', \mathbf{T})$ is surjective and it is obviously injective so that $G' \longrightarrow G$ is a weak isomorphism. ∎

3. Weak and strong topologies

3.1. Weak topologies. We will say that a space is **weakly topologized** if it has the coarsest possible topology for its set of characters.

3.2. Definition. *If G and G' belong to $\mathcal{G}$ a not necessarily continuous homomorphism $f : G' \longrightarrow G$ will be called* **weakly continuous** *if composition with f induces a map $X(G) \longrightarrow X(G')$. This means that whenever $\chi : G \longrightarrow \mathbf{T}$ is continuous, so is $\chi . f : G' \longrightarrow \mathbf{T}$. If f is also bijective, we will say it is a* **weak isomorphism***.*

Let $G \in \mathcal{G}$. We will denote by σG the group σ elements as G, topologized as a subgroup of $\mathbf{T}^{X(G)}$. It is clear that σG has the same characters as G with a weaker topology so that the identity map $G \longrightarrow \sigma G$ is continuous and a weak isomorphism. Moreover, the homomorphism $f : G' \longrightarrow G$ is weakly continuous if and only if it induces a continuous map $\sigma G' \longrightarrow \sigma G$. When $G \longrightarrow \sigma G$ is a topological isomorphism, we say that G has the **weak topology**. Let $\mathcal{W}$ denote the full subcategory of $\mathcal{G}$ of the weakly topologized groups.

3.3. Proposition. *The functor σ is a coreflection of the inclusion of $\mathcal{W}$ into $\mathcal{G}$. It preserves finite products.*

Proof. The first sentence is obvious. As for the second, we can write $\sigma = IC$, where $C : \mathcal{G} \longrightarrow \mathcal{W}$ is left adjoint to the inclusion $I : \mathcal{W} \longrightarrow \mathcal{C}$. Since both categories are additive, finite products and sums coincide. Since left adjoints preserve sums and right adjoints preserve products, we have

$$\begin{aligned} \sigma(G \times H) &= IC(G \times H) = IC(G \oplus H) \cong I(CG \oplus CH) \\ &\cong I(CG \times CH) \cong ICG \times ICH = \sigma G \times \sigma H \end{aligned}$$ ■

3.4. Proposition. *A group in $\mathcal{G}$ has the weak topology if and only if it can be embedded in a compact group.*

Proof. One way follows immediately from the definition. Suppose there is an embedding $G \subseteq K$, where K is compact. This gives rise to a surjection $X(K) \twoheadrightarrow X(G)$. Pontrjagin duality implies that $K \longrightarrow \mathbf{T}^{X(K)}$ is a topological embedding. In the commutative diagram

$$\begin{array}{ccc} G & \hookrightarrow & K \\ \downarrow & & \downarrow \\ \mathbf{T}^{X(G)} & \longrightarrow & \mathbf{T}^{X(K)} \end{array}$$

the fact that the upper and right hand maps are topological embeddings implies that the left hand arrow is. ■

3.5. Proposition. *Suppose $f : G \longrightarrow G'$ is a weak isomorphism and G' can be topologically embedded in a compact group. Then f is continuous and induces an isomorphism $\sigma G \longrightarrow G'$.*

Proof. Since $X(G') \cong X(G)$, we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & G' \\ \downarrow & & \downarrow \\ \mathbf{T}^{X(G)} & \xrightarrow[\cong]{} & \mathbf{T}^{X(G')} \end{array}$$

Since f followed by an embedding is continuous, so is f. Moreover, up to isomorphism, G' has the topology induced on G by the inclusion $G \subseteq \mathbf{T}^{X(G)}$. ■

3.6. Corollary. *For any groups G and G', we have $\sigma G \times \sigma G' \cong \sigma(G \times G')$.*

Proof. Evidently $X(G \times G') \cong X(G) \times X(G')$, while $\sigma(G \times G')$ is a subgroup of a compact group. ■

We note that the discrete group $\mathbf{Z}$ of integers is not weakly topologized. In fact, no compact group can contain an infinite discrete group. For if $\mathbf{Z} \subseteq K$, a compact group, the fact that $\mathbf{Z}$ is discrete implies there is a neighbourhood U of 0 in K with $U \cap \mathbf{Z} = \{0\}$. If V is a neighbourhood of 0 with $V + V \subseteq U$, then it is immediate that when K is covered by translates of V no one of them can contain more than one element of $\mathbf{Z}$ and therefore cannot have a finite refinement.

3.7. Strong topologies. We will say that a space is **strongly topologized** if it has the finest possible topology for its set of characters.

3.8. Theorem. *Every locally compact group has a strong topology.*

Proof. Let G be locally compact. We must show that the identity map $G \longrightarrow \tau G$ is continuous. Since $\tau G \in \mathcal{G}$, it has an embedding $\tau G \hookrightarrow \prod G_i$ with each G_i locally compact. Each map $\tau G \longrightarrow G_i$ induces $X(G_i) \longrightarrow X(\tau G) = X(G)$. Then by [Glicksberg, 1962, Theorem 1.1], the map $G \longrightarrow G_i$ is continuous and thus $G \longrightarrow \tau G$ is. ∎

We denote by $\mathcal{S}$ and $\mathcal{W}$ the full subcategories of $\mathcal{G}$ consisting of the strongly and weakly topologized groups, respectively.

4. $*$-autonomous categories and the chu construction

4.1. $*$-autonomous categories. Although the definition we give here is not the most general, it is appropriate for this note. A $*$-autonomous category is a category $\mathcal{C}$ with the following structures:

1. For every pair of objects A and B an object $A \multimap B$, thought of as an internal version of the set of morphisms of $A \longrightarrow B$;
2. For every pair of objects A and B, a tensor product $A \otimes B$;
3. A unit object $\mathbf{Z}$
4. A dualizing object $\mathbf{T}$.

These are subject to various identities, of which the most important are natural isomorphisms

1. $\mathrm{Hom}(\mathbf{Z}, A \multimap B) \cong \mathrm{Hom}(A, B)$;
2. $A \otimes Z \cong A$;
3. $A \otimes B \cong B \otimes A$;
4. $\mathrm{Hom}(A, B \multimap C) \cong \mathrm{Hom}(A \otimes B, C)$;
5. $A \cong ((A \multimap \mathbf{T}) \multimap T)$.

The last isomorphism comes from the identity map under the isomorphisms

$$\mathrm{Hom}(A \multimap \mathbf{T}, A \multimap \mathbf{T}) \cong \mathrm{Hom}((A \multimap T) \otimes A, \mathbf{T}) \cong \mathrm{Hom}(A \otimes (A \multimap \mathbf{T}), \mathbf{T})$$
$$\cong \mathrm{Hom}(A, (A \multimap \mathbf{T}) \multimap \mathbf{T})$$

There are other "coherence laws" that will not be mentioned exlicitly here because the $*$-autonomous categories considered here will have $A \multimap B$ as the set $\mathrm{Hom}(A, B)$ with structure making it an object of the ambient category. Under those circumstances, the coherence is automatic.

We denote the dual object $A \multimap \mathbf{T}$ by A^*.

4.2. The chu construction. Let $\mathcal{Ab}$ denote the category of discrete abelian groups and T denote the discrete circle group. We construct a $*$-autonomous category called $\mathrm{chu}(\mathcal{Ab}, T)$. An object is a pair (A, X) together with a pairing $\langle -, - \rangle : A \otimes X \longrightarrow T$ such that for each $a \in A$ there is an $x \in X$ with $\langle a, x \rangle \neq 0$ and for each $x \in X$ there is an $a \in A$ with $\langle a, x \rangle \neq 0$. A map $f : (A, X) \longrightarrow (B, Y)$ is a homomorphism $f : A \longrightarrow B$ such that for each $y \in Y$ there is an $x \in X$ such that for all $a \in A$, we have $\langle f(a), y \rangle = \langle a, x \rangle$.

Here is a different way of looking at this. We see that in any pair (A, X), we can think of X as a separating subgroup of $\mathrm{Hom}(A, T)$ and then the condition is that for any $y \in Y$, we have that $y.f \in X$.

Given objects (A, X) and (B, Y) we define $(A, X) \multimap (B, Y) = (\mathrm{Hom}((A, X), (B, Y)), A \otimes Y/t(A, Y))$ where $t(A, Y)$ is the subgroup of $A \otimes B$ required for the separation condition. First we describe the pairing. For $f : (A, X) \longrightarrow (B, Y)$ and $a \otimes y \in A \otimes Y$, we define $\langle f, a \otimes y \rangle = \langle fa, y \rangle$ and extended linearly to all of $A \otimes Y$. If $f \neq 0$, then there is an $a \in A$ with $f(a) \neq 0$ and then there is a $y \in Y$ with $\langle f(a), y \rangle \neq 0$ so that $\langle f, a \otimes y \rangle \neq 0$ and the separation condition is satisfied. On the other hand there could well be elements of $A \otimes Y$ on which $\langle f, - \rangle$ vanishes for all $f : (A, X) \longrightarrow (B, Y)$ and we let $t(A, Y)$ denote the subgroup of all such elements.

Finally, we let the dualizing object be T.

4.3. Theorem. *The category* $\mathrm{chu}(\mathcal{Ab}, T)$ *is* $*$*-autonomous.*

The proof is somewhat tedious, although not at all hard, and can be found in a number of places. We omit it. Naturally, $(A, X)^* = (X, A)$ with the same pairing and the second dual isomorphism is the identity.

5. The equivalences

5.1. Theorem. *The category of weak spaces is equivalent to* $\mathrm{chu}(\mathcal{Ab}, \mathbf{T})$.

Proof. We define a functor $F : \mathcal{W} \longrightarrow \mathrm{chu}(\mathcal{Ab}, \mathbf{T})$ by $F(G) = (|G|, X(G))$. The pairing is evaluation. We let $R : \mathrm{chu}(\mathcal{Ab}, \mathbf{T}) \longrightarrow \mathcal{W}$ by $R(A, X)$ is the abelian group A topologized as a subgroup of $\mathbf{T}^X$. Then we claim that R is right adjoint to F, that $FR \cong \mathrm{Id}$ and that when G is weak, then the comparison map $G \longrightarrow RF(G)$ is an isomorphism.

If $f : F(G) = (|G|, X(G)) \longrightarrow (A, X)$ is given, then for any $\varphi \in X$, $\varphi.f \in X(G)$. Thus the composite $G \longrightarrow R(A, X) \longrightarrow \mathbf{T}$ is continuous for all such φ, so that $G \longrightarrow R(A, X) \longrightarrow \mathbf{T}^X$ is continuous. But $R(A, X)$ is topologized as a subspace of $\mathbf{T}^X$ and so $f : G \longrightarrow R(A, X)$ is continuous. The uniqueness of f is obvious, which establishes the adjunction.

To make the second claim, we must show that the group of characters on $R(A, X)$ is X. Any character on $R(A, X)$ extends, by uniform continuity of continuous homomorphisms and the fact that $\mathbf{T}$ is compact, hence complete, to $\mathbf{T}^X$. The map $A \longrightarrow \mathbf{T}^X$ factors through the group $\mathrm{Hom}(X, \mathbf{T})$ and the group of continuous characters on the latter is just X.

As for the third claim, the definition of weak is such that G is weak if and only if G is topologized as a subgroup of $\mathbf{T}^{X(G)}$. ∎

5.2. Theorem. *The category of strong spaces is equivalent to* $\mathrm{chu}(\mathcal{Ab}, \mathbf{T})$.

Proof. We use F as above and define $L : \mathrm{chu}(\mathcal{Ab}, \mathbf{T}) \longrightarrow \mathcal{S}$ by $L(A, X) = \tau R(A, X)$. If $f : (A, X) \longrightarrow F(G) = (|G|, X(G))$ is given, apply R to get $f : R(A, X) \longrightarrow RF(G)$ and the latter is readily seen to be $\sigma(G)$. Now apply $\tau = \tau\sigma$ to get $L(A, X) \longrightarrow \tau(G)$ which, together with the canonical $\tau(G) \longrightarrow G$ gives $L(A, X) \longrightarrow G$. The uniqueness is clear.

Since $F\tau = F$, from the definition of F, we have$FL(A, X) = F\tau R(A, X) = FR(A, X) \cong (A, X)$, we see that $FL \cong \mathrm{Id}$. If G has the strong topology, then the fact that $LF(G) \longrightarrow G$ is a bijection that induces the identity on the character groups implies it is an isomorphism. ∎

5.3. Theorem. *The dual of a locally compact group is the same as the Pontrjagin–van-Kampen dual.*

Proof. Let L be locally compact and $\widehat{L}$ denote its Pontrjagin–van-Kampen dual. Then we know that $\widehat{L}$ is locally compact and, so by 3.8, it has the strong topology. But the duality $X(\widehat{L}) = |L|$ and the same is true of L^*. Since L^* and $\widehat{L}$ have the same underlying abelian group, the same characters, and the strong topology, they are the same. ∎

6. The $*$-autonomous structure

It follows from the results of the previous section that $\mathcal{S}$ and $\mathcal{W}$ are equivalent to each other and to $\mathrm{chu}(\mathcal{Ab}, \mathbf{T})$. Since the last is $*$-autonomous, so are the other two. Here we will give explicit descriptions of the structure.

We first describe the structure on $\mathcal{S}$. If G, $H \in \mathcal{S}$, the underlying group of $G \multimap H$ is $\mathrm{Hom}(G, H)$. The topology is the strong topology for the group $G \otimes X(H)$ that takes the element $x \otimes \chi$ to the character on $\mathrm{Hom}(G, H)$ defined by $f \mapsto \chi(f(x))$. This is a separating set of characters, since when $f \neq 0$, there is some $x \in G$ for which $f(x) \neq 0$ and then a character $\chi \in X(H)$ for which $\chi(f(x)) \neq 0$. For the tensor product, we begin with the algebraic tensor product $|G| \otimes |H|$. Each $f \in Hom(G, H^*)$ determines a character on $|G| \otimes |H|$ by the formula $f(x \otimes y) = f(x)(y)$. This set of functionals does not, in general, separate the points of $|G| \otimes |H|$ but if we factor the elements that are annihilated by every element of $\mathrm{Hom}(G,H)$, we get a separating set on the quotient and then define $G\otimes H$ as this quotient with the strong topology induced by the set of characters.

The structure on $\mathcal{W}$ is similar. Just use the weak topology on $G \multimap H$ and $G\otimes H$ instead of the strong.

We end with an example to show that it is possible that $G\otimes H=0$ even when $|G|\otimes|H|$ is not. The tensor product $|\mathbf{T}|\otimes|\mathbf{T}|$ consists of a direct sum of $2^{\aleph_0}$ copies of $\mathbf{Q}$. But $\mathrm{Hom}(\mathbf{T},\mathbf{T}^*)=\mathrm{Hom}(\mathbf{T},\mathbf{Z})=0$ so the only character on $|\mathbf{T}|\otimes|\mathbf{T}|$ is 0 and so $\mathbf{T}\otimes\mathbf{T}=0$. This is not surprising in light of the facts that $\mathbf{Q}/\mathbf{Z}\otimes\mathbf{Q}/\mathbf{Z}=0$ as ordinary groups and $\mathbf{Q}/\mathbf{Z}$ is dense in $\mathbf{R}/\mathbf{Z}=\mathbf{T}$.

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