Iterated Galois connections in arithmetic and linguistics

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Abstract: Galois connections may be viewed as pairs of adjoint functors, specialized from categories to partially ordered sets. We study situations that permit iterations of such adjoints. While their occurrence in elementary number theory is a curiosity, they play a crucial rôle in a new algebraic approach to sentence structure in natural languages.

1. Adjunction and complementation.

We begin by looking at two well-known functions in number theory:

$p(n) =$ the n^{th} prime (for good measure, we put $p(0) = 0$),

$\pi(n) =$ the number of primes $\leq n$.

Inspection of the following table leads to a curious observation first made in [1]:

n	$p(n)$	$p(n)+n$	$\pi(n)$	$\pi(n)+n+1$
0	0	0	0	1
1	2	3	0	2
2	3	5	1	4
3	5	8	2	6
4	7	11	2	7
5	11	16	3	9
...	...	...	...	...

We note that the sets

$$\{p(n)+n | n \in \mathbf{N}\}, \qquad \{\pi(n)+n+1 | n \in \mathbf{N}\}$$

are complementary subsets of $\mathbf{N}$. The proof, though tricky, is quite easy. It has nothing whatever to do with properties of prime numbers and depends only on the fact that p and π constitute a Galois correspondence:

$$p(x) \leq y \;\Leftrightarrow\; x \leq \pi(y).$$

We borrow the terminology of category theory and call p the *left adjoint* of π.

In general, let $f, g : \mathbf{N} \to \mathbf{N}$ be order preserving functions such that

$$f(x) \leq y \;\Leftrightarrow\; x \leq g(y),$$

then

$$\begin{aligned} f(x) + x \leq x + y \quad &\Leftrightarrow\; x + y \leq g(y) + y \\ &\Leftrightarrow\; g(y) + y + 1 > x + y. \end{aligned}$$

It follows immediately that the sets

$$\text{(i)} \qquad F = \{f(x) + x | x \in \mathbf{N}\},\ G + 1 = \{g(y) + y + 1 | y \in \mathbf{N}\}$$

have no elements in common. Moreover, consider the range

$$0 \leq x \leq n, \qquad 0 \leq y \leq n - x.$$

Then F and $G+1$ together have exactly $n+1$ elements between 0 and n, hence all the elements between 0 and n. Thus F and $G+1$ are complementary sets.

We leave as an exercise the converse observation: if F and G are infinite subsets of $\mathbf{N}$ so that F and $G+1$ are complementary (hence $0 \in F$), then F and G have the form (i) with f left adjoint to g.

2. Iterated adjoints.

When does an order preserving function $g : \mathbf{N} \to \mathbf{N}$ have a left adjoint?

More generally, let

$$f : (X, \leq) \to (Y, \leq), \quad g : (Y, \leq) \to (X, \leq)$$

be a Galois connection, that is

$$\text{(ii)} \qquad f(x) \leq y \ \Leftrightarrow \ x \leq g(y)$$

This equivalence may also be written:

$$\text{(iii)} \qquad f(x) = \inf\{y \in Y | x \leq g(y)\}.$$

Given g order preserving, we can find its left adjoint $f = g^\ell$ if and only if

(a) g preserves infs,

(b) the inf (iii) exists.

This is a special case of Freyd's Adjoint Functor Theorem in category theory.

Return now to the special case $X = Y = \mathbf{N}$. Then

$$f(x) = \min\{y \in \mathbf{N} | x \leq g(y)\}.$$

In this case, (a) holds trivially and (b) holds if and only if $\{y \in \mathbf{N} | x \leq g(y)\} \neq \emptyset$ for each x in $\mathbf{N}$, that is, provided g is unbounded.

We note that then $f = g^\ell$ will also be unbounded, since $f(x) \leq b$ would imply $x \leq g(b)$. Therefore, g^ℓ also has a left adjoint $g^{\ell\ell}$, and so on.

In summary, if $g : \mathbf{N} \to \mathbf{N}$ has a left adjoint g^ℓ, then it also has iterated left adjoints $g^{\ell\ell}$, $g^{\ell\ell\ell}$ etc.

One way to construct these is to look at the corresponding subsets of $\mathbf{N}$:

The functions

$$g,\ g^\ell,\ g^{\ell\ell},\ \cdots$$

correspond to the sets

$$G,\ (G+1)^c,\ ((G+1)^c+1)^c,\ \cdots,$$

where F^c denotes the complement of F.

What about the right adjoints? (ii) may also be written thus:

$$\text{(iv)} \qquad g(y) = \sup\{x \in X | f(x) \leq y\}.$$

Given f, its right adjoint $g = f^r$ exists if and only if

(a$'$) f preserves sups,

(b$'$) the sup (iv) exists.

In the special case $X = Y = \mathbf{N}$, sup $=$ max, hence

(a$'$) holds if and only if $f(0) = 0$ (the supremum of the empty set being 0),

(b$'$) holds if and only if f is unbounded.

Thus the order preserving function $f : \mathbf{N} \to \mathbf{N}$ has a right adjoint f^r if and only if $f(0) = 0$ and f is unbounded.

What about iterated right adjoints?

$$f,\ f^r,\ f^{rr},\ \cdots$$

correspond to the sets

$$F,\ F^c - 1,\ (F^c-1)^c - 1,\ \cdots .$$

Now

$$\begin{aligned} f^r(0) = 0 \text{ iff } 0 \in F^c - 1,&\ \text{i.e. } 1 \in F^c,\ \text{i.e. } 1 \notin F, \\ f^{rr}(0) = 0 \text{ iff } 0 \in (F^c-1)^c - 1,&\ \text{i.e. } 1 \in (F^c-1)^c, \\ \text{i.e. } 1 \notin F^c - 1,&\ \text{i.e. } 2 \notin F^c,\ \text{i.e. } 2 \in F. \end{aligned}$$

Continuing in this way, we see that iterated right adjoints exists if and only if $F = \{f(x) + x | x \in \mathbf{N}\}$ is the set of *even* numbers, i.e. $f(x) + x = 2x$, i.e., f is the *identity* function.

For example, if $f = p$, $p^r = \pi$ and $p^{rr} = \pi^r$, where $\pi^r(y) = p(y+1) - 1$. But $\pi^r(0) = 1 \neq 0$, so $p^{rrr} = \pi^{rr}$ does not exist.

However, let $f : \mathbf{Z} \to \mathbf{Z}$ be order preserving and unbounded on *both* sides. Then f is left adjoint to g if and only if

$$F = \{f(x) + x | x \in \mathbf{Z}\},\ G + 1 = \{g(y) + y + 1 | y \in \mathbf{Z}\}$$

are complementary subsets of $\mathbf{Z}$ (see [2]). In this case all iterated left and right adjoints exist, as was pointed out in [3]. Note that, in general, f^ℓ and f^r are distinct. For example, if $f(x) = 2x$, then $f^r(x) = [x/2]$ and $f^\ell(x) = [(x+1)/2]$.

3. Adjunction in $2-$categories.

Adjoints are usually defined in the $2-$category of all small categories, where

$$\begin{aligned} 0 - \text{cells} &= \text{small categories,} \\ 1 - \text{cells} &= \text{functors,} \\ 2 - \text{cells} &= \text{natural transformations.} \end{aligned}$$

The usual definition carries over to *any* $2-$category: let $f : A \to B$ and $g : B \to A$ be $1-$cells, then f is left adjoint to g if and only if there exist $2-$cells

$$\eta : 1_A \to gf, \qquad \varepsilon : fg \to 1_B$$

such that

$$(v) \qquad g \to gfg \to g = 1_g, \quad f \to fgf \to f = 1_f.$$

As perhaps the simplest example of a $2-$category, let us look at any *partially ordered monoid*, where

$$\begin{aligned} 0 - \text{cells} &: \text{just one,} \\ 1 - \text{cells} &= \text{elements,} \\ 2 - \text{cells} &: f \leq g. \end{aligned}$$

(Note that there is at most one 2-cell between two 1-cells and we write $f \leq g$ for $f \to g$.) Here elements f and g form an adjoint pair if and only if

$$1 \leq gf, \quad fg \leq 1.$$

The equations (v), like other equations between $2-$cells, are automatically satisfied.

As an example consider the partially ordered monoid of all order preserving functions $\mathsf{N} \to \mathsf{N}$ under composition. Then f is left adjoint to g provided

$$f(x) \leq y \;\Leftrightarrow\; x \leq g(y).$$

4. Pregroups.

We shall introduce a couple of definitions.

A *left pregroup* is a partially ordered monoid in which every element has a left adjoint. Of course, every partially ordered group is a left pregroup in which $f^\ell = f^{-1} = f^r$. More interesting is the partially ordered monoid of all unbounded order preserving functions $\mathsf{N} \to \mathsf{N}$.

A *right pregroup* is a partially ordered monoid in which every element has a right adjoint. Finally, a *pregroup* is both a left and a right pregroup. Again, every partially ordered group is a pregroup, but so is the partially ordered monoid of all order preserving functions $\mathsf{Z} \to \mathsf{Z}$ which are unbounded on both sides. In this example, $f^\ell \neq f^r$ in general.

Recent applications to linguistics make use of the *free pregroup* generated by a partially ordered set of so-called *basic types*. Given a basic type a, one forms *simple types*

$$\cdots, a^{\ell\ell},\ a^{\ell},\ a,\ a^{r},\ a^{rr},\ \cdots\ .$$

Compound types, or just *types*, are strings of simple types, say

$$A = \alpha_1 \alpha_2 \cdots \alpha_n\ ,$$

the α_i being simple types. In particular, when $n = 0$ one obtains the empty string 1.

The types from a monoid under concatenation which is partially ordered due to the order of basic types and the following *contractions*

$$U\beta^{\ell}\alpha V \leq UV,\ U\alpha\beta^{r}V \leq UV \text{ if } \alpha \leq \beta$$

and *expansions*

$$UV \leq U\beta\alpha^{\ell}V, \quad UV \leq U\alpha^{r}\beta V \text{ if } \alpha \leq \beta.$$

Both contractions and expansions are needed to prove that the compound types form a pregroup with adjoints

$$A^{\ell} = \alpha_n^{\ell} \cdots \alpha_1^{\ell}, \quad A^{r} = \alpha_n^{r} \cdots \alpha_1^{r}\ .$$

But there is a kind of Church-Rosser theorem [3], which asserts:

Without loss in generality, one may assume that all contractions precede all expansions.

It follows that to show $A \leq \beta$, where β is a simple type, no expansions are needed. In particular, to show that a string of words of compound type A is a sentence of type s in a natural language it suffices to check that $A \leq s$ by repeated contractions.

5. Linguistic applications.

We shall discuss briefly how free pregroups may help to investigate certain aspects of three European languages:

(1) Chomskyan traces in English [3,4],
(2) word order in German [5],
(3) clitic pronouns in French [6].

It so happens that the same list of basic types will do for the fragments of the languages discussed here.

$\pi_j =$ j-th person pronoun ($j = 1, 2, 3$),
$s_1 =$ declarative sentence in present tense,
$o =$ direct object,
$p_2 =$ past participle,
$q_1 =$ yes-or-no question in present tense,
$q =$ yes-or-no question, $\quad q_1 \leq q$,
$w =$ wh-question,
$i =$ infinitive of intransitive verb.

English.

I see her

$$\pi_1\ (\pi_1^r s_1 o^\ell)\ o\ \le s_1$$

I have seen her

$$\pi_1\ (\pi_1^r s_1 p_2^\ell)(p_2 o^\ell) o\ \le s_1$$

have I seen her?

$$(q_1 p_2^\ell \pi_1^\ell)\pi_1\ (p_2\ o^\ell)\ o\ \le\ q_1$$

whom have I seen − ?

$$(w o^{\ell\ell} q^\ell)(q_1\ p_2^\ell \pi_1^\ell)\pi_1\ (p_2\ o^\ell)\ \le\ w$$

Note that $q^\ell q_1 \le q^\ell q \le 1$. The dash here represents a Chomskyan trace, which is put in for comparison only. In writing *whom* rather than *who*, I am following the late Inspector Morse.

she is seen −

$$\pi_3\ (\pi_3^r s_1 o^{\ell\ell} p_2^\ell)(p_2\ o^\ell)\ \le\ s_1$$

she has been seen −

$$\pi_3\ (\pi_3^r s_1 p_2^\ell)(p_2\ o^{\ell\ell} p_2^\ell)(p_2\ o^\ell)\ \le\ s_1$$

German.

du siehst ihn

$$\pi_2\ (\pi_2^r\ s_1 o^\ell)\ o\quad \le s_1$$

siehst du ihn ?

$$(q_1 o^\ell \pi_2^\ell)\ (\pi_2\ \ o\quad \le q_1$$

ich habe ihn gesehn

$$\pi_1\ (\pi_1^r\ s_1 p_2^\ell)\ o\ (o^r p_2)\quad \le s_1$$

ich kann ihn sehen

$$\pi_1\ (\pi_1^r s_1 i^\ell)\ o\ (o^r i)\quad \le s_1$$

er kann gesehn werden

$$\pi_3\ (\pi_3^r\ s_1 i^\ell)\ (o^r p_2)\ (p_2^r o^{rr} i)\quad \le s_1$$

$$\begin{array}{c} \textit{kann er gesehn werden ?} \\ (q_1 i^\ell \pi_3^\ell)\ \pi_3\ (o^r p_2)\ (p_2^r o^{rr} i) \quad \leq q_1 \end{array}$$

French.

$$\begin{array}{c} \textit{je veux dormir} \\ \pi_1\ (\pi^r{}_1\ s_1 i^\ell)\ i \quad \leq s_1 \end{array}$$

$$\begin{array}{c} \textit{je veux voir Jean} \\ \pi_1\ (\pi^r{}_1\ s_1 i^\ell)\ (i o^\ell)\ o \quad \leq s_1 \end{array}$$

$$\begin{array}{c} \textit{je veux le voir} \\ \pi_1\ (\pi^r{}_1\ s_1 i^\ell)\ (i o^{\ell\ell} i^\ell)(i o^\ell) \quad \leq s_1 \end{array}$$

Note that the three languages considered here all require the simple type $o^{\ell\ell}$, but German also requires o^{rr}. So far, I have not come across a language that requires $o^{\ell\ell\ell}$ or o^{rrr}.

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