QUATERNIONS AND THREE TEMPORAL DIMENSIONS

JOACHIM LAMBEK

Abstract. The application of quaternions to special relativity predicts a six-dimensional universe, which uncannily resembles ours, except that it admits three dimensions of time. Yet its mathematical description with the help of quaternions gains in transparency, due to the crucial observation that every skew-symmetric four-by-four real matrix is the sum of two matrices representing multiplication by vector quaternions on the left and on the right respectively.

1. Historical introduction

Among mathematicians and theoretical physicists there have been a dedicated few, albeit outside the mainstream, who attempted to formulate the laws of physics in the language of quaternions, motivated by four-dimensionality common to the world we live in and the division ring of quaternions. Pursuing the quaternionic description of special relativity, I came to the counter-intuitive conclusion that this approach really suggests that nature ought to have six dimensions: three of space and three of time. Many other theories also predict more than four dimensions, e.g. Kaluza-Klein requires five and string theory ten, but usually additional spatial ones.

Only a few years after the advent of special relativity, it was realized that this theory could be formalized with the help of *biquaternions*, that is, quaternions with complex components [Conway 1911, 1912; Silverstein 1912, 1924]. The main idea was to represent location in space-time by the *Hermitian* biquaternion

$$i(i_1x_1 + i_2x_2 + i_3x_3) + t$$

where i_1, i_2 and i_3 are the quaternion units and i is the imaginary square root of -1, or equivalently by

$$i_1x + i_2x_2 + i_3x_3 + it,$$

which will be the starting point of the present investigation.

The quaternion units here can be replaced by their regular matrix representations, either as two-by-two complex matrices or as four-by-four real ones, which will be favoured here. When looking at the Dirac equation, it was found convenient to replace the complex number i by a matrix representing multiplication on the right, say by i_1, as well [Conway 1951, Gürsey 1955, Lanczos 1929]. But why not i_2 or i_3?

I have mentioned only a few articles relevant to the present discussion. There are other important publications employing quaternions for treating modern physics, e.g. Finkelstein et al [1979], who looked at the connection with quantum logic, Adler [1995], who investigated quaternionic Hilbert space, and Baez and Huerta [2009, 2010], who employed quaternions and other division algebras as leading to supersymmetry.

2. Mathematical prerequisites

A quaternion has the form

$$a = a_0 + i_1a_1 + i_2a_2 + i_3a_3,$$

where the a_α are real numbers and the quaternion units i_α satisfy

$$i_1^2 = i_2^2 = i_3^2 = i_1i_2i_3 = -1.$$

The quaternions form a *division algebra*: each non-zero quaternion has an inverse $\overline{a}/N(a)$, where

$$\overline{a} = a_0 - i_1a_1 - i_2a_2 - i_3a_3$$

is the *conjugate* of a and

$$N(a) = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

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is its *norm*. Quaternions may be represented by 2×2 complex matrices or by 4×4 real ones. Here we will confine attention to the latter and single out the left and right *regular* representatives $L(a)$ and $R(a)$.

With any quaternion ψ there is associated a four-vector $(\psi_0, \psi_1, \psi_2, \psi_3)$, which we will turn into a column vector denoted by $[\psi]$. One defines the matrices $L(a)$ and $R(a)$ by

$$L(a)[\psi] = [a\psi],\ R(a)[\psi] = [\psi a]$$

and notes that

$$L(aa') = L(a)L(a'),\ R(bb') = R(b')R(b),\ L(a)R(b) = R(b)L(a).$$

A well-known theorem about central simple algebras implies in particular that the tensor product of the quaternion algebra with its opposite is the algebra of all 4×4 real matrices. Computationally, this means that each such matrix A can be written uniquely as a linear combination of the $L(i_\alpha)R(i_\beta)$, where α and β range from 0 to 3 and $i_0 = 1$. Thus

$$A = \sum_{\alpha=0}^{3} L(i_\alpha)R(a_\alpha) = \sum_{\beta=0}^{3} R(i_\beta)L(a'_\beta),$$

where a_α and a'_β are quaternions. Exploiting the quaternion conjugate, we may define

$$A^* = \sum L(i_\alpha)R(\overline{a}_\alpha),\ A^\dagger = \sum R(i_\beta)L(\overline{a}'_\beta) \tag{2.1}$$

and verify that

$$A^{*\dagger} = A^{\dagger *} = A^T,$$

the *transposed* matrix of A.

For a skew-symmetric matrix A, the transposed $A^T = -A$, which equation may be written as $A^\dagger = -A^*$, hence

$$A = L(\mathbf{a}) + R(\mathbf{a}'),$$

for some *vector* quaternions $\mathbf{a}$ and $\mathbf{a}'$, whose scalar parts $a_0 = 0$ and $a'_0 = 0$.

The representations L and R are not unrelated. Let Γ be the diagonal matrix with entries $(1, -1, -1, -1)$, then

$$\Gamma[\psi] = [\overline{\psi}],$$

hence

$$\Gamma L(\overline{a})\Gamma[\psi] = \Gamma L(\overline{a})[\overline{\psi}] = \Gamma[\overline{a}\overline{\psi}] = \Gamma[\overline{\psi a}] = [\psi a] = R(a)[\psi].$$

Thus

$$R(a) = \Gamma L(\overline{a})\Gamma$$

and similarly

$$L(a) = \Gamma R(\overline{a})\Gamma,$$

hence

$$\Gamma(L(\mathbf{a}) + R(\mathbf{a}')\Gamma = -R(\mathbf{a}) - L(\mathbf{a}').$$

Most of the discussion that follows employs arguments that are well-known in the familiar four dimensions and indeed reduce to the latter if two of the time dimensions are deleted.

3. Six-dimensional Lorentz transformations

We will assume that space-time consists of the skew-symmetric matrices

$$X = L(\mathbf{x}) + R(\mathbf{t}),$$

where

$$\mathbf{x} = i_1x_1 + i_2x_2 + i_3x_3,\ \mathbf{t} = i_1t_1 + i_2t_2 + i_3t_3$$

represent the spatial and temporal position respectively (although usually $\mathbf{t} = it$). We write

$$X^\dagger = -L(x) + R(t),\ X^* = L(x) - R(t),$$

hence

$$\begin{aligned} XX^* &= L(\mathbf{x}^2) - R(\mathbf{t}^2) \\ &= -L(\mathbf{x} \circ \mathbf{x}) + R(\mathbf{t} \circ \mathbf{t}) \\ &= (-\mathbf{x} \circ \mathbf{x} + \mathbf{t} \circ \mathbf{t})I \\ &= -X \odot X. \end{aligned}$$

Explicit mention of the identity matrix will usually be suppressed.[1)]

A four-by-four matrix Q of determinant 1 will give rise to a *Lorentz transformation*

$$X \mapsto QXQ^T$$

provided

$$(QXQ^T)^* = Q^{-T}X^*Q^{-1}, \tag{3.1}$$

hence

$$X^* \mapsto Q^{-T}X^*Q^{-1}.$$

It follows that the quadratic form XX^* is invariant, since the scalar XX^* commutes with the matrix Q. The stipulation that the determinant of Q is 1 ensures that the transformation $X \mapsto QXQ^T$ does not change the determinant of X and also that $Q = \Gamma$ is excluded, so Lorentz transformations cannot interchange space and time.

The Lorentz transformations form a group.[2)] For, if Q and Q' satisfy (3.1), so do $Q'Q$ and Q^{-1}. Indeed,

$$\begin{aligned} ((Q'Q)X(Q'Q)^T)^* &= (Q'QXQ^TQ'^T)^* \\ &= Q'^{-T}(QXQ^T)^*Q'^{-1} \\ &= Q'^{-T}Q^{-T}X^*Q^{-1}Q'^{-1} \\ &= (Q'Q)^{-T}X^*(Q'Q)^{-1}. \end{aligned}$$

Moreover, replacing X by $Q^{-1}XQ^{-T}$ in (3.1), we obtain

$$X^* = Q^{-T}(Q^{-1}XQ^{-T})^*Q^{-1},$$

hence

$$(Q^{-1}XQ^{-T})^* = Q^TX^*Q.$$

As we shall see, a number of other skew matrices S transform like X, so SS^* will be invariant as well.

Conversely, the invariance of XX^* implies (3.1), as is easily shown.

4. The Lorentz category

It appears that all interesting physical entities in special relativity are representable as four-by-four real matrices accompanied by instructions on how they are transformed by a Lorentz transformation. Thus

$$A \mapsto Q^uAQ^{-v},$$

where Q^{-v} is short for $(Q^v)^{-1}$ and u and v are elements of the commutative monoid $\{0, 1, \#\}$, with $Q^0 = I,\ Q^1 = Q$ and $Q^\# = Q^{-T} = (Q^T\}^{-1}$.

Note that

$$u0 = 0,\ u1 = u,\ \#\# = 1.$$

We will write $A : u \to v$ and consider this to be an arrow in the *Lorentz category.*

The Lorentz category is an additive category, which Barry Mitchell would have called a *ring with three objects.* In fact, when disregarding quantum mechanics, we do not require the object 0. An additive category with two objects is known as a *Morita context* to ring theorists.[3)]

When $A : u \to v$ and $B : v \to w$, then $AB : u \to w$, since

$$AB \mapsto Q^uAQ^{-v}Q^vBQ^{-w} = Q^uABQ^{-w}.$$

AA does not make sense, unless $u = v$, but AA^* does, as long as A transforms like X, for then

$$A \mapsto QAQ^\#,\ A^* \mapsto Q^\#A^*Q^{-1},$$

which means that

$$A : 1 \to \#,\ A^* : \# \to 1,\ AA^* : 1 \to 1.$$

We may think of the skew matrices $A : 1 \to \#$ as *basic* arrows in the Lorentz category.

If A and B are skew-symmetric matrices with $A : 1 \to \#$ and $B : 1 \to \#$, we have $AB^* : 1 \to 1$, but also

$$\frac{1}{2}(AB^* \pm BA^*) : 1 \to 1.$$

Let

$$A = L(\mathbf{a}) + R(\mathbf{a}'),\ B^* = L(\mathbf{b}) - R(\mathbf{b}'),$$

then the *scalar* part of AB^* is

$$\frac{1}{2}(AB^* + BA^*) = -\mathbf{a} \circ \mathbf{b} + \mathbf{b}' \circ \mathbf{a}' = -A \odot B,$$

say, thus extending the Heaviside scalar product to skew-symmetric matrices. On the other hand,

$$\frac{1}{2}(AB^* - BA^*) = L(\mathbf{a} \times \mathbf{b}) + R(\mathbf{a}' \times \mathbf{b}') + L(\mathbf{b})R(\mathbf{a}') - L(\mathbf{a})R(\mathbf{b}').$$

This is a bit more complicated than the familiar vector product of two quaternionic three-vectors.

Now consider a third skew-symmetric matrix $C : 1 \to \#$, then

$$AB^*C : 1 \to \#$$

as well, but this is not in general skew-symmetric. However, one calculates

$$AB^*C + CB^*A = A(B^*C + C^*B) + (AB^* + BA^*)C - (AC^* + CA^*)B.$$

This is twice the skew-symmetric part of AB^*C, hence

$$\text{Skew } (AB^*C) = -A(B \odot C) + B(C \odot A) - C(A \odot B). \tag{4.1}$$

5. Special relativity in six dimensions

Most, if not all physical quantities in special relativity may be represented by basic arrows $S : 1 \to \#$ and their starred conjugates $S^* : \# \to 1$ by composition. To start with, there are the basic arrows $S = X$ and $S = P$, where

$$X = L(\mathbf{x}) + R(\mathbf{t})$$

represents locations in *space-time* and

$$P = L(\mathbf{p}) + R(\mathbf{m})$$

represents *energy-momentum*. Note that $\mathbf{m}$ is a three-vector generalizing the usual

$$\text{mass} = \text{energy} = 4\pi \times \text{ frequency}$$

(according to Einstein and de Broglie).

From $X : 1 \to \#$ and $P^* : \# \to 1$ we obtain $XP^* : 1 \to 1$, which breaks up into the scalar

$$\frac{1}{2}(XP^* + PX^*) = -L(\mathbf{x} \circ \mathbf{p}) + R(\mathbf{t} \circ \mathbf{m}) = -X \odot P,$$

generalizing the four-dimensional $-\mathbf{x} \circ \mathbf{p} + tm$, and the anti-scalar

$$\begin{aligned} \frac{1}{2} \quad & (XP^* - PX^*) = XP^* + X \odot P \\ = \quad & L(\mathbf{x} \times \mathbf{p}) + R(\mathbf{t} \times \mathbf{m}) - L(\mathbf{x})R(\mathbf{m}) + R(\mathbf{t})L(\mathbf{p}), \end{aligned}$$

which generalizes the four-dimensional

$$\mathbf{x} \times \mathbf{p} - i(\mathbf{x}m - t\mathbf{p}).$$

Physicists require that, for all genuine particles ("virtual" particles excepted), $X \odot X \leq 0$ and $P \odot P \leq 0$, and they define the *interval* s and the rest-mass μ, by

$$X \odot X = -s^2, \;\; P \odot P = -\mu^2.$$

For particles with non-zero rest-mass $P = \mu dX/ds$ and so

$$dX/ds \odot dX/ds = -1, \;\; dX/ds \odot P = -\mu.$$

Force $d\mathbf{p}/ds$ and power $d\mathbf{m}/ds$ may then be combined into dP/ds, the rate of change of energy-momentum.

To X and P we must add two arrows $1 \to \#$:

$$J = L(\mathbf{J}) + R(\boldsymbol{\rho}), \;\; A = L(\mathbf{A} + R(\boldsymbol{\phi})),$$

extending the usual *charge-current density* and *four-potential* to six dimensions.

At this point, I should mention that I have chosen to render the velocity c of light, the Dirac form $h/2\pi$ of Planck's constant and the dielectric constant ϵ in vacuum all equal to 1.

To the above skew-symmetric matrices $1 \to \#$ we may add the partial differential operator

$$D = L(\nabla) - R(\nabla') : 1 \to \#,$$

where

$$\nabla' = i_1 \frac{\partial}{\partial t_1} + i_2 \frac{\partial}{\partial t_2} + i_3 \frac{\partial}{\partial t_3},$$

such that

$$D[\psi] = [\overrightarrow{\nabla}\psi - \psi\overleftarrow{\nabla'}],$$

it now being assumed that $\psi = \psi(X)$ is a quaternion depending on the location X in space-time. The minus sign here is due to the fact that $\partial/\partial x_\alpha$ has the opposite variance from dx_α, as illustrated by the observations that

$$d = dx \odot D = d\mathbf{x} \circ \nabla + dt \circ \nabla'.$$

6. The Maxwell and Lorentz equations

Traditionally, one would express the motion of a charged particle with electric charge q by the equation

$$\frac{d}{ds}(P + qA) = 0,$$

where qA is the *potential* energy-momentum.

In our familiar four-dimensional universe,

$$\begin{aligned} D^*A = & \left(i\frac{\partial}{\partial t} + \nabla\right)(i\phi + \mathbf{A}) \\ = & -\frac{\partial \phi}{\partial t} - \nabla \circ \mathbf{A} + i\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\phi\right) + \nabla \times \mathbf{A} \\ = & -\left(\frac{\partial \phi}{\partial t} + \nabla \circ \mathbf{A}\right) - i\mathbf{E} + \mathbf{B}, \end{aligned}$$

where $\mathbf{E}$ and $\mathbf{B}$ denote the electric and magnetic field respectively. It follows that

$$\frac{1}{2}(D^*A - A^*D) = -\left(\frac{\partial \phi}{\partial t} + \nabla \circ \mathbf{A}\right)$$

and

$$\frac{1}{2}(D^*A + A^*D) = \mathbf{B} - i\mathbf{E}.$$

In the six-dimensional universe we would therefore require that

$$\begin{aligned} D^*A &= (L(\nabla) + R(\nabla'))(L(\mathbf{A}) + R(\boldsymbol{\phi})) \\ &= -D \odot A + B - E, \end{aligned}$$

where

$$D \odot A = L(\nabla \circ \mathbf{A}) + R(\nabla' \circ \boldsymbol{\phi})$$

is the scalar part of D^*A. Moreover,

$$B = L(\nabla \times \mathbf{A}) + R(\nabla' \times \boldsymbol{\phi})$$

and

$$-E = L(\nabla)R(\boldsymbol{\phi}) + R(\nabla')L(\mathbf{A})$$

will yield what might be called the "vector part"

$$\frac{1}{2}(D^*A - A^*D) = D^*A + D \odot A = B - E = F,$$

where F denotes the electro-magnetic field.

As far as F is concerned, A is determined only up to a *gauge transformation*

$$A \mapsto A - D\Theta = A_\Theta,$$

when Θ is a scalar, since $\frac{1}{2}(D^*A - A^*D)$ is then diminished by the vector part of $D^*D\Theta$, which is zero.

This observation is traditionally exploited by assuming that $D \odot A = 0$. When calculating the potential energy-momentum of a charged particle, I prefer to ensure instead that

$$\frac{dX}{ds} \odot A_\Theta = 0.$$

This easily follows, provided we postulate that

$$\frac{d\Theta}{ds} = \frac{dX}{ds} \odot A.$$

To obtain the special relativistic version of the *Lorentz force*, we apply equation (4.1) with

$$A = q\frac{dX}{ds},\ B = D,\ C = A_\Theta$$

and obtain

$$\text{Skew}\ \left(q\frac{dX}{ds}D^* A_\Theta\right) = -q\frac{dX}{ds}(D \odot A_\Theta) + D\left(A_\Theta \odot q\frac{dX}{ds}\right) - A_\Theta\left(q\frac{dX}{ds} \odot \overleftarrow{D}\right).$$

Here the second term is zero, the third is $-\frac{d}{ds}qA_\Theta$ and the first may be transposed to the left, yielding

$$\text{Skew}\ \left(q\frac{dX}{ds}(D^* A_\Theta + D \odot A_\Theta)\right) = -\frac{d}{ds}qA_\Theta,$$

hence

$$dP/ds = \text{ Skew}\ \left(q\frac{dX}{ds}F\right) = -\frac{d}{ds}qA_\theta.$$

This is the relativistic version of the Lorentz force + power and ensures that

$$\frac{d}{ds}(P + qA_\theta) = 0,$$

the conservation of energy-momentum.

In our familiar four-dimensional universe, we have the relativistic equation

$$\begin{aligned} dP/ds = \text{ Skew}\ \left(q\frac{dX}{ds}F\right) &= \text{ Skew}\left(q\frac{dt}{ds}(i + \mathbf{v})(\mathbf{B} - i\mathbf{E})\right) \\ &= q\frac{dt}{ds}(\mathbf{v} \times \mathbf{B} + \mathbf{E} + i\mathbf{v} \circ \mathbf{E}), \end{aligned}$$

where $q(\mathbf{v} \times \mathbf{B} + \mathbf{E})$ is usually called the *Lorentz force* and $q(\mathbf{v} \circ \mathbf{E})$ might be called the *Lorentz power.*

The electro-magnetic field F allows us to calculate not only the Lorentz force acting on a charged particle, but also the charge current density that gives rise to it by Maxwell's equations. In the six-dimensional universe, these may be replaced by the single equation

$$DF - J = 0,$$

that is

$$(L(\nabla) - R(\nabla'))(B - E) = L(\mathbf{J}) + R(\boldsymbol{\rho}),$$

where

$$B = L(\nabla \times \mathbf{A}) + R(\nabla' \times \boldsymbol{\phi}) = L(\mathbf{B}) + R(\mathbf{B}')$$

and

$$-E = L(\nabla)R(\boldsymbol{\phi}) + R(\nabla')L(\mathbf{A}).$$

Note that in our four-dimensional universe

$$\mathbf{B}' = 0,\ R(\boldsymbol{\phi}) = i\phi,\ R(\nabla') = i\frac{\partial}{\partial t}$$

hence we get the usual Maxwell equations, combining those of Faraday, Gauss and Ampère, as well as the equation asserting the non-existence of a magnetic monopole.

The six-dimensional *equation of continuity* still takes the form

$$D \odot J = 0. \tag{6.1}$$

It holds because the scalar part of $D^*J = D^*DF$ is zero, since the scalar part of F is zero.

7. The Dirac equation

To reconcile quantum mechanics with special relativity one begins with the so-called *Klein-Gordon* equation, already known to Schrödinger. In six dimensions this takes the form:

$$DD^*\Psi = -\mu^2\Psi.$$

For the moment, we will allow Ψ to be any four-by-four matrix whose entries are real valued functions of X. If $\mu \neq 0$, this can evidently be replaced by the pair of first order equations

$$D^*\Psi_1 = \mu\Psi_2^*,\ D\Psi_2^* = -\mu\Psi_1,$$

where both Ψ_1 and Ψ_2 satisfy the Klein-Gordon equation. According to Penrose [2004], they can be viewed as representing the *Zig* and the *Zag* of a moving particle with nonzero rest-mass. Dirac went one step further and replaced them by a single first order equation.

This can be done quite easily in our present setup if we assume the existence of a skew matrix $K : 1 \to \#$ such that

$$K \odot K = -1,\ K \odot D = 0,\ K \odot P = 0,$$

that is,

$$KK^* = 1,\ KD^* + DK^* = 0,\ KP^* + PK^* = 0.$$

This assumption is justified if there is a coordinate system in which one of the time dimensions, say t_3, is irrelevant, that is, the action takes place in a five-dimensional subspace of the six-dimensional space under consideration. Then it follows that $\partial\Psi/\partial t_3 = 0$ and we may take $K = R(i_3)$ in this frame of reference.

Putting

$$\Psi = \Psi_1 - K\Psi_2^*,$$

we calculate

$$\begin{aligned} D^*\Psi = &\ D^*\Psi_1 + K^*D\Psi_2^* \\ = &\ \mu\Psi_2^* - K^*\mu\Psi_1 \\ = &\ -\mu K^*(\Psi_1 - K\Psi_2^*) \\ = &\ -\mu K^*\Psi. \end{aligned}$$

This is the *Dirac equation.*

Our argument works for $\Psi : 1 \to u$, u being any object of the Lorentz category, but Dirac assumed that $u = 0$ and called Ψ a *spinor*. Then Ψ may be multiplied by any constant matrix on the right, e.g. by the column matrix

$$[1] = (1000)^T,$$

yielding $\Psi = [\psi]$, where ψ is a quaternion. The Dirac equation in quaternionic form becomes

$$\vec{\nabla}\psi + \psi\overleftarrow{\nabla}' + \mu(\mathbf{k}\psi - \psi\mathbf{k}') = 0,$$

where

$$K = L(\mathbf{k}) + R(\mathbf{k}').$$

Making use of the explicit solution

$$\Psi_1(X) = \cos(X \odot P)\Psi(0)$$

so that

$$\Psi_2^*(X) = -\mu^{-1}P^*\sin(X \odot P)\Psi(0),$$

we obtain the explicit solution of the Dirac equation:

$$\Psi(X) = (\cos(X \odot P) + \mu^{-1}KP^*\sin(X \odot P))\Psi(0).$$

Now let

$$\mu\eta = KP^* = -PK^*,$$

then $\eta^2 = -1$, hence this can be written more elegantly:

$$\Psi(X) = \exp(\eta(X \odot P))\Psi(0).$$

There is no reason why Dirac's equation should not be valid in a six-dimensional universe outside a five-dimensional subspace, even if it is not provably equivalent to the Klein-Gordon equation. One argument in favour of this is that, multiplying the Dirac equation by

$$\eta' = \mu^{-1}K^*P = -\mu^{-1}P^*K,$$

transforms it into

$$\eta' D^*[\psi] = P^*[\psi].$$

This shows that P^* is an eigenvalue of $\eta' D^*$, when η' now plays the rôle of the imaginary square root of -1 in the early non-relativistic quantum mechanics.

8. Epilogue

Although quaternions had been expected to describe our $3+1$-dimensional space-time, they fit more elegantly into a $3+3$-dimensional one. All the arguments applying quaternions to modern physics presented here reduce to familiar arguments when two of the time dimensions are suppressed, yet they appear more natural in the wider setting. One is reminded of Plato's cave with the suggestion that we mortals see only the four-dimensional shadow of the outer reality of six dimensions.

What then can one make of the extra dimensions? The least we can say is that our traditional space-time is a subspace of the $3+3$-dimensional one. Which particular subspace may be left to a choice of coordinate system with the help of an appropriate Lorentz transformation. At any rate, I feel that further study deserves to be devoted to the representations of the $3+3$-dimensional orthogonal group that leaves the quadratic form

$$x_1^2 + x_2^2 + x_3^2 - t_1^2 - t_2^2 - t_3^2$$

invariant and happens to be a subgroup of the group of linear transformations of four-dimensional real vector space. Finally, the additional dimensions of time make it easier to understand how Schrödinger's cat can be alive and dead at the same time t, if t^2 is interpreted to be $t_1^2 + t_2^2 + t_3^2$.

Traditionally, space-time is represented by a 4-vector and the electro-magnetic field by a 6-vector, aka a skew-symmetric matrix. When we assume that time is three-dimensional, it is space-time that is represented by a skew-symmetric matrix and the Maxwell field will acquire additional terms, hopefully encoding at least the weak force. Repeating this process in curved space-time should also account for the gravitational force.

When I presented these ideas at a mathematical seminar, a remarkable document by Peter Gillan [2010] was brought to my attention. Gillan exploits the three-dimensionality of time to incorporate the Standard Model into *general* relativity. He seems to account for all the forces of nature and even manages to calculate the rest-masses of all the elementary particles, thus going way beyond what I have suggested here, and appears to achieve the grand unification that Einstein dreamt about. I can only hope that my modest quaternionic approach may serve as an elementary introduction to his profound ideas.[4)]

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Endnotes

1) Here the small circle is used for the traditional Heaviside scalar product and $\odot$ stands for its generalization to skew-symmetric matrices.

2) It has been suggested that this group is related to groups familiar to physicists such as SO(3,3) and SO(4). If so, I leave this to the experienced reader.

3) The arrow $1 \to 0$ stands for the transformation $\Psi \mapsto Q\Psi$, which already in four-dimensional space-time applies to a Dirac spinor, as in my 2013 article (in which categories were avoided).

4) On the other hand, he has been criticized for not believing in the Higgs particle, whose existence seems to be confirmed by a recent discovery at CERN.

Department of Mathematical & Statistical Sciences, McGill University, Montreal, Quebec