Capulet semantics and the prehistory of mathematics and science

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1 Introduction

Philosophically inclined linguists tend to favour Montague semantics, a brave attempt to translate natural language into an intensional variant of the language of set theory.[1)]

By "Capulet semantics" I have in mind the study of the meaning of words not already encompassed by Montague semantics, not excluding etymological and other historical side-excursions. I believe that natural languages embody a kind of folk philosophy of their speakers, which anticipates more mature forms of mathematics and science. I am even tempted to say that much philosophical speculation is influenced by the philosophers' introspection into their own language.

In order to provide the present enquiry with focus, let me look at one particular English sentence[2)]:

(1) *My uncle drank a glass of water.*

While this sentence is innocent enough, a deeper look at the words contained in it will reveal some interesting problems and help to throw light on what I see as the prehistory of mathematics and various sciences, both natural and social.

To warm up, let us recall that English nouns, like the nouns in many languages, are divided into count nouns and mass nouns. The former can be preceded by an indefinite article, the latter cannot. The former usually possess a plural form, the latter do not. Compare count nouns *pig, bean* etc with mass nouns *pork, rice* etc. However, categories can change in time or in context. The noun *pea* is today a count noun, but is derived by back-formation from an old mass noun *pease*. *Man* is normally a count noun, but it can become a mass noun in the mouth of a cannibal who prefers man to pork. *Beer* is normally a mass noun, but it can become a count noun when you order two beers.

2 The subject

Let us look at the noun phrase *my uncle*, which occurs as the subject of (1). It means " X's uncle" or "the uncle of X", where X is the present speaker.

For the moment, take a closer look at the first person possessive pronoun *my*. What is remarkable about the first letter m is that it represents the first person in many languages and provides evidence for their common origin. Thus, in many Indo-European and Ural-Altaic languages, m is attached to the verb in the first person. While, in English verbs, it has only survived in *am*, in Latin verbs it denotes the first person in the imperfect past tense. In Polish, Hungarian and Turkish, it is used more widely. The fact that m tends to denote the speaker is one clue used by Greenberg [1996] and Ruhlen [1994] to establish the Eurasiatic language super-family that stretches from the Atlantic to the Pacific.

Next, look at the word *uncle*. On the face of it, it is a count noun, having a plural. What interests us here, however, is that it usually requires a compliment, as in

(2) $$Joe's\ uncle \ = \ the\ uncle\ of\ Joe.$$

In fact, the word *uncle* denotes a binary relation. Its primary meaning is of course "parent's male sibling", symbolically PSM. Here P and S are more basic binary relations "parent" and "sibling". Moreover, M is a subrelation of the identity relation, when this is restricted to males, and juxtaposition denotes the relative product.

More precisely,

$$bPSMa \text{ means } \exists_x \exists_y (bPy \wedge ySx \wedge xMa),$$

when xMa means that $x = a$ is a male. Such expressions belong to what may well be the earliest form of mathematics[3)], the calculus of relations. As Chomsky [1979] pointed out: "the Greeks made up number theory, others made up kinship systems."

The word *uncle* also has a secondary meaning, referring to the male spouse of a parent's sibling, symbolically $PS\Sigma M$, where Σ denotes the relation "spouse". This basic relation depends on the sociological context of the linguistic community and has recently acquired some ambiguity in societies that allow same-sex marriages. Anyway, we can show that

$$PS\Sigma M \to PSM$$

if we adopt the rewrite rule $S\Sigma \to S$, which allows a sibling's spouse to be called a sibling. Such rewrite rules are in fact quite common as we shall see.[4)]

Of course, the word *uncle* can also have a number of tertiary meanings, as "mother's boyfriend" or as in "Uncle Sam", but these we will ignore. Instead, we will take a look at the etymology of *uncle*. This word is derived from Latin *avunculus*, meaning "little grandfather". In ancient Rome, a woman was supposed to have a male legal guardian. If her father died, her brother took over this rôle. Not surprisingly, our word *nephew* is derived from an older word for "grandson", as in Sanskrit *naptr*. In modern Italian, the word *nepote* is ambiguous, meaning either "grandson" or "nephew". Anyway, we can express these facts by the rewrite rules

$$PFSM \to PPM, \qquad SFC \to CC$$

for Latin kinship terminology, where C stands for "child", the converse of "parent", and F denotes identity of females.

We should point out that Romans distinguished between the mother's brother and the father's brother:

$$PMSM \to \ patruus \ \neq \ avunculus.$$

English too, at one time, distinguished these two relations. Thus, old English had

$$PMSM \to \ foedera, \qquad PFSM \to \ \bar{e}am.$$

The word *uncle* is a relatively new addition to the vocabulary. In German, the word *Onkel* was borrowed even more recently from French and replaced the earlier forms

$$PMSM \to Vetter, \qquad PFSM \to \ Oheim,$$

although the former survives only with the meaning "male cousin".

For comparison, in the Trobriand islands,

$$PMSM \to PM \to tama, \qquad PFSM \to \ kada,$$

according to data collected by Malinowski [1932], the former being explained with the help of a rewrite rule postulated by Lounsbury [1965]. However, there are five different words for cousin, depending on the genders G_i in $G_1PG_2SG_3CC_4$, one of which is again *tama*:

$$MPMSFCM \ \to \ MPMPFCM \ \to \ MPMSM \ \to \ tama,$$

using Lounsbury's rewrite rules

$$PMSF \ \to \ PMPF, \qquad PFC \ \to \ S,$$

and one of the cousins is not considered to be a relative at all, since PMC is undefined! See [BL1995] for more details.

3 The verb

Next, let us look at the finite verb form *drank*, the past tense of *drink /drinks*. The English verb has two simple tenses, present and past, or four simple tenses if we count the almost obsolete subjunctive. Compare this with French, with seven simple tenses, although two of them, the definite past and the past subjunctive are in the process of disappearing. The distinction between past and present is by no means universal; for example, Arabic distinguishes between complete and incomplete actions instead.

On the face of it, *to drink* means to consume a liquid and is to be distinguished from *to eat*, supposedly restricted to the consumption of solids. When an Englishman eats tea, he may be consuming cucumber sandwiches; when he drinks soup, he is indeed consuming a liquid. But why do Americans eat soup?

Surprisingly, modern science tells us that glass is a liquid, albeit a very slowly flowing one. In principle, my uncle could drink glass, but this is not what the sentence (1) asserts, in view of the innocent little word *a*.

4 The object

Let us now look at the noun phrase *a glass of water*. Here the indefinite article *a* assures us that the glass in question is not the substance glass, but a container like a cup.

The indefinite article is usually translated by the existential quantities in elementary logic courses, yet a word of caution is necessary. In

(3) *if a man drinks water, he will stay sober*

the indefinite article must be translated by a universal quantifier.

Whereas water cannot be counted, glasses of water can. Whereas a substance can only be measured, the units of measure may be counted. Already the ancient Greeks were concerned with the contradiction between what can be counted and what must be measured, between the discrete and the continuous (see [AL1995]). To resolve this contradiction, Aristotle proposed a distinction between matter and form. In "glass of water", water is the matter and glass is the form.[5)] Similarly, we have "loaf of bread", "cake of soap", "head of cattle", and so on in English. Note, however, that Indonesians count cattle by their tails.

Already the pre-Socratic philosophers were debating whether the world is made up of continuous substances, which can be infinitely divided without losing their identity, or whether the world consists of discrete objects; whether it is more important to measure, as in geometry, or to count, as in arithmetic.

Thales proclaimed that everything is made from a single substance, namely water. Not surprisingly, he is also the first to be remembered for making significant contributions to geometry, both pure and applied. Some of his followers preferred different primitive substances, until a final consensus proposed four elementary substances: earth, water, air and fire,[6)] reminiscent of the modern three states of matter (solid, liquid and gas) together with energy, into which matter is now known to be transformable.

Other Greek philosophers, primarily Democritus, conjectured that substances are not really infinitely divisible, but are made up of indivisible units, called "atoms". The legendary Pythagoras went even further, saying that numbers, meaning positive integers, themselves were the ultimate constituents. His school was more interested in arithmetic than in geometry. A crisis arose when it was discovered that the ratio of the diagonal of a square to its side could not be expressed as the ratio of two positive integers. The ingenious proof of this first appears in the works of Aristotle (see the [1985] translation) and was only later added to revised versions of Euclid's Elements (see the [1956] translation). Aristotle used it to illustrate the type of argument known as "reductio ad absurdum".

Appendix I. Mathematics at Plato's Academy

Much of the mathematics at Plato's Academy was concerned with the problem of how to resolve the contradiction between geometry and arithmetic, between ratios of geometric quantities, which we call positive real numbers, and ratios of positive integers, which we call rational numbers. Plato's disciples gave two answers.

According to Eudoxus, two ratios of geometric quantities, say α/β and γ/δ, are equal provided, for all positive integers p and q,

$$q\alpha \geq p\beta \text{ if and only if } q\gamma \geq p\delta,$$

and similarly with $\geq$ replaced with $\leq$. This definition is presented in Euclid's Elements and is equivalent to the modern definition of Dedekind (see e.g. the [1963] edition), according to which $\alpha/\beta = \gamma/\delta$ means that

$$\alpha/\beta \geq p/q \text{ if and only if } \gamma/\delta \geq p/q,$$

and similarly with $\geq$ replaced with $\leq$.

On the other hand, Theaetetus is now known to have discovered continued fractions, called "anthyphareises" by Aristotle, according to which

$$\sqrt{2} = 1 + \frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\cdots$$

When this potentially infinite expression is terminated after the first, second, third and fourth step, we obtain the following approximation to $\sqrt{2}$, already known to the Pythagoreans:

$$1,\ 3/2,\ 7/5,\ 17/12.$$

More spectacularly, the golden ratio

$$(\sqrt{5}-1)/2 = \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots$$

plays a crucial rôle in Euclid's Elements, where its construction precedes that of the regular pentagon.

Continued fractions were forgotten and only rediscovered in the seventeenth century. However, as was shown by Fowler [1987], they had played an integral part in Plato's Academy. More recently, Stelios Negrepontis [2002] observed that continued fractions metaphorically enter all of Plato's dialectical dialogues (see the [1989] translation).

One problem with continued fractions is the absence of an algorithm for adding two such. Therefore people now prefer the more general Cauchy sequences: each equivalence class of positive Cauchy sequences contains exactly one continued fraction as a representative member. Whereas the Eudoxus-Dedekind construction of real numbers is equivalent to the Theaetetus-Cauchy construction as far as classical mathematics is concerned, they offer demonstrably different approaches from an intuitionistic point of view. (See e.g. Johnstone [1977].)

Appendix II. Relations in modern mathematics

Binary relations, as distinct from functions, did not really attract the attention of mathematicians until the nineteenth century, and then only marginally. They were advocated by Peirce (see his collective works of [1931-58]) and Schroeder [1890-1905], whose interest was more philosophical than algebraic. Relations only entered algebra at the end of the century in a little known paper by Goursat [1989], which was not as influential as it ought to have been.

A slightly updated version of Goursat's result concerns a *homomorphic* relation R between two groups G and H, that is a binary relation whose graph is a subgroup of $G \times H$. This gives rise to an isomorphism between two other groups obtained from $RR^\vee$ and $R^\vee R$ viewed as congruence relations on their domain, $R^\vee$ being the converse of R. It took another half century for Goursat's theorem to be applied to the Zassenhaus Lemma [La1957], crucial in refining composition series of groups, by letting R be the relative product of two congruence relations on subgroups of G and H respectively.

It must be admitted that, even today, the calculus of relations has not yet entered the mathematical mainstream, in spite of valiant attempts by Tarski [1987] and Freyd [1990].

Appendix III. Fundamental particles today

Is the world we live in made up of substances, which can be infinitely divided but not counted, or of particles which can be counted but not subdivided?

Most modern physicists still side with Zeno that time and space are infinitely divisible and with Democritus that matter (and energy) consists of indivisible units. Of course, the nature of these units has evolved with the progress of science. In the nineteenth century, an apparently indivisible unit of water turned out to consist of two atoms of hydrogen linked to one atom of oxygen. The apparently indivisible atoms were seen, in the early twentieth century, to consist of *electrons, protons* and *neutrons*. More recently, the last two of these were decomposed into *quarks*.

At the latest count, the generally accepted fundamental particles were *fermions* and *bosons*. In the so-called Standard Model, the fermions are electrons, positrons, neutrinos and quarks of three different colours. All these come in three generations or families, of which we shall concentrate on the first. The bosons are *photons, gluons* and carriers of the weak force. There is still no consensus on the nature of the *graviton*, the alleged carrier of the gravitational force, which Einstein saw as curvature of space-time.

Modifying the views of Harari [1979,1983] and Shupe [1979], I figure [La2000] that the generally accepted fundamental particles can be represented by four-vectors (a_0, a_1, a_2, a_3) with entries $a_i = 0, 1$ or -1, and that all known interactions between two such particles giving rise to a third can be described by adding the four-vectors. In particular, $a_0 = 0$ for bosons, $a_0 = 1$ for fermions and $a_0 = -1$ for their anti-particles. Moreover, $a_1 + a_2 + a_3$ is $3/e$ times the electric charge, taking the charge of the electron to be $-e$.

For example,

electron :	$(1, -1, -1, -1)$
neutrino :	$(1, 0, 0, 0)$
W^- :	$(0, -1, -1, -1)$
photon, Z° :	$(0, 0, 0, 0)$
red upquark :	$(1, 0, 1, 1)$
red downquark :	$(1, -1, 0, 0)$
blue upquark :	$(1, 1, 0, 1)$
blue downquark :	$(1, 0, 0 - 1, 0)$
red-blue gluon :	$(0, 1, -1, 0)$

A sample Feynman diagram would show that an electron may turn into a neutrino and a W^-, while an upquark plus the W^- turns into a downquark. Here Z° and W^- are bosons which carry the so-called weak force. (See Feynman [1985] and Gell-Mann [1994].)

$$\begin{array}{lcccr} \text{neutrino} & \nwarrow & & \nearrow & \text{red downquark} \\ & & \overset{W^-}{\cdots\cdots\rightarrow} & & \\ \text{electron} & \nearrow & & \nwarrow & \text{red upquark} \end{array}$$

Similarly, the following diagram shows how a gluon mediates the strong force between quarks:

red downquark ↖		↗ red downquark
	red-blue gluon · · · · · · · · · · · · →	
blue upquark ↗		↖ blue downquark

It must be admitted that the four-vector representation has some shortcomings. The photon and the Z° are represented by the same four-vector. Two out of eight alleged gluons are not represented at all[6] and there is no account of the higher generation fermions.

Altogether, there are 81 possible four-vectors, only 25 of which represent known particles: eight first generation fermions, eight anti-fermions and nine gauge bosons.

Of course, adding or subtracting two of the 25 four-vectors may yield a four-vector which does not have all its entries equal to 0, 1 or -1. Let us admit only those additions and subtractions which do. We note that the four-vectors representing colourless fundamental particles, with $a_1 = a_2 = a_3$, are closed under admissible additions and subtractions. However, the closure under addition and subtraction of the set of all four-vectors representing known fundamental particles consists of all 81 four-vectors.

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FOOTNOTES

[1] To a mathematician this looks like a functor transforming the structure of sentences into morphisms of a cartesian closed category.

[2] I have used the same sentence in [1995] to illustrate the nominalistic interpretation of natural languages. Here it is to serve a different purpose, but some overlap with the earlier paper is inevitable.

[3] Evidently, kinship relations were not designed by modern mathematicians. They would have taken S to be a reflexive relation, making a person his own sibling.

[4] In some societies, though not in ours, there is a rewrite rule $S\Sigma \to \Sigma$.

[5] Aristotle was a sexist; he said that, in procreation, women provide the matter and men the form.

[6] For a more recent exponent of this view see [Br2001].

[7] I would have suggested that the weak vector bosons W^+ and W^-, which may transform upquarks into downquarks of the same colour, might do double duty as gluons, but this is not the accepted view, according to which these would be represented by the same four-vector as the photon and the Z°.