ON $*$-AUTONOMOUS CATEGORIES OF TOPOLOGICAL MODULES

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Abstract. Let R be a commutative ring whose complete ring of quotients is R-injective. We show that the category of topological R-modules contains a full subcategory that is $*$-autonomous using R itself as dualizing object. In order to do this, we develop a new variation on the category $\mathrm{chu}(\mathcal{D}, R)$, where $\mathcal{D}$ is the category of discrete R-modules: the high wide subcategory, which we show equivalent to the category of reflexive topological modules.

1. Introduction

Let R be a ring (with unit). In [Barr, et. al. (2009)], we showed that under certain reasonable conditions on R, a full subcategory of the category of right R-modules is equivalent to a full subcategory of the category of topological left R-modules. These conditions are explained in detail in the cited paper. When R is commutative, the conditions simplify and reduce to the assumption that the complete ring of quotients Q of R, as described in [Lambek (1986), Section 3] be R-injective (*op. cit.,* Proposition 4.3.3). It is sufficient, but far from necessary, that R have no non-zero nilpotents. When K is a field and $p \in K[x]$ is any polynomial, the residue ring $R = K[x]/(p)$ will always be its own complete ring of quotients and also be self-injective. But if p has repeated factors, R will have nilpotents.

The main purpose of this paper is to show that when the commutative ring R satisfies the injectivity condition of the preceding paragraph, then the category of topological R-modules contains a full subcategory with both an autoduality and an internal hom. Such a category is called $*$-autonomous, see, for example, [Barr (1999)]. As usually happens, exhibiting such a structure requires a detour through the Chu construction (*op. cit.*). However, since we are not supposing that R is self-injective, neither the full Chu category nor the separated extension subcategory does the job and we are forced to introduce yet another variation of the Chu category, that we call the **high, wide** subcategory (Definition 4.8).

We also consider the case that R is self-injective. Then all Chu objects are high and wide and we do not need to introduce that subcategory; moreover, there will be two distinct $*$-autonomous subcategories of topological R-modules. Although the two categories are equivalent as categories—even as $*$-autonomous categories—one of them contains all the discrete modules and the other one doesn't.

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2. The ring R

All objects we study in this note are modules over a commutative ring R of which we make one further assumption: that the complete ring of quotients of R be R-injective (for which it is necessary and sufficient that the complete ring of quotients be self-injective).

An ideal I of a commutative ring R is called **dense** if whenever $0 \neq r \in R$, then $rI \neq 0$. The complete ring of quotients Q of R is characterized by the fact that it is an essential extension of R and every homomorphism from a dense ideal to Q can be extended to a homomorphism $R \longrightarrow Q$. Details are found in [Lambek (1986), Sections 2.3 and 4.3].

It is worth going into a bit more detail about the reference [Lambek (1986)]. In section 2.3, the complete ring of quotients for commutative rings is constructed, while in 4.3 the construction is carried out in the non-commutative case, where the definition of "dense" is more complicated. Unfortunately, all the discussion of injectivity is carried out in the latter section and it is not easy to work out reasonable conditions under which the complete ring of quotients is injective. Here is one simple case, although far from the only one.

2.1. Example. Let K be a field. Each of the rings $K[x]/(x)^2$ and $K[x, y]/(x, y)^2$, can be readily seen to be its own complete rings of quotients (they have no proper dense ideals), but the first is and the second is not self-injective. In the second case the ideal (x, y) contains every proper ideal (and is therefore large, as defined in the third paragraph below), but is not dense.

Let A be an R-module. An element $a \in A$ is called a **weak torsion element** if there is a dense ideal $I \subseteq R$ with $aI = 0$. We say that A is **weak torsion module** if every element of A is and that A is **weak torsion free** if it contains no non-zero weak torsion elements.

An R-module is said to be **R-cogenerated** if it can be embedded into a power of R. A topological R-module is called **R-cogenerated** if it can be embedded algebraically and topologically into a power of R, with R topologized discretely. Among other things, this implies that the topology is generated by (translates of) the open submodules.

An ideal $I \subseteq R$ is called **large** if its intersection with every non-zero ideal is non-zero. An obvious Zorn's lemma argument shows that if every map from a large ideal of R to a module Q can be extended to a map $R \longrightarrow Q$, then this is true for all ideals and so Q is R-injective. A dense ideal is characterized by the fact that its *product* with every non-zero ideal is non-zero. If, for example, there are no nilpotents, then when the product of two ideals is non-zero, so is their intersection and then the complete ring of quotients is injective. But while this condition is sufficient, it is far from necessary as the example $K[x]/(x)^2$ makes clear.

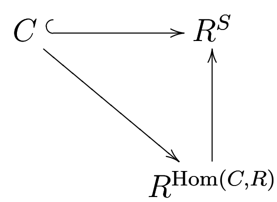
3. The category $\mathcal{C}$

Let $\mathcal{C}$ denote the category of R-cogenerated topological R-modules. When C is an object of $\mathcal{C}$, we let $|C|$ denote the discrete module underlying C and let $\|C\|$ denote the discrete set underlying C. If C and C' are objects of $\mathcal{C}$, we let $\hom(C, C')$ denote the R-module of continuous R-linear homomorphisms and $\mathrm{Hom}(C, C')$ denote the set $\|\hom(C, C')\|$.

We denote by $C^\star$ the module $\hom(C, R)$ topologized as a subspace of $R^{\|C\|}$.

3.1. Proposition. *For $C \in \mathcal{C}$, the canonical map $C \longrightarrow R^{\|C^\star\|}$ is a topological embedding.*

Proof. Let $C \hookrightarrow R^S$ define the topology on C. Then for each $s \in S$, the composite $C \longrightarrow R^S \xrightarrow{p_s} R$, where p_s is the projection, defines a homomorphism $C \longrightarrow R$. Thus there is a function $S \longrightarrow \mathrm{Hom}(R, C)$ that leads to the commutative triangle



and the diagonal map, being an initial factor of a topological embedding, is one itself. ■

For later use, we prove a theorem of a purely topological nature. It is true for arbitrary topological R-modules, not just for objects of $\mathcal{C}$.

3.2. Theorem. *Suppose $A \subseteq \prod_{s\in S} A_s$ is an embedding of topological modules and D is a discrete module. Then every continuous map $A \longrightarrow D$ factors through a quotient of A of the form $A/(A \cap (\prod_{s\in S-S_0}))A_s$ where S_0 is a finite subset of S.*

Proof. If $f : A \longrightarrow D$ is given, then $\ker f$ must be open. A subbasic open neighbourhood of 0 in the product consists of a set of the form $A \cap \left(U \times \prod_{s\neq s_1} A_s\right)$ with U open in A_{s_1}. In particular, every subbasic open neighbourhood at 0 includes a set of the form $A \cap \left(\prod_{s\neq s_1} A_s\right)$ and every basic neighbourhood of 0, being a finite intersection of subbasic sets, contains a set of the form $A \cap (\prod_{s\notin S_0} A_s)$ which implies that for some choice of a finite set S_0, f vanishes on $A \cap (\prod_{s\in S_0} A_s)$. ■

In fact, by analyzing this argument we find that $\hom(-, D)$ commutes with filtered colimits (a product is a filtered colimit of the finite products) whenever D contains a neighbourhood of 0 that contains no non-zero submodule. However, we require only the form stated here.

3.3. Corollary. *For any family $\{A_s\}$ of topoogical modules and any discrete object D, we have that the canonical map $\sum \hom(A_s, R) \longrightarrow \hom(\prod A_s, D)$ is an isomorphism.* ■

Crucial to this paper is the following theorem, which is proved in [Barr, et. al. (2009), Corollary 3.8]. It is understood that R always carries the discrete topology. Incidentally, this is the only place that the injectivity condition is used.

3.4. THEOREM. *Let $C \hookrightarrow C'$ be an algebraic and topological inclusion between objects of $\mathcal{C}$. Then the cokernel of $\hom(C', R) \longrightarrow \hom(C, R)$ is weak torsion.* ∎

4. The chu category

We begin this section with a brief description of the Chu categories and chu categories that concern us in this article. They are instances of a much more general construction. For more details, see [Barr (1998), Barr (1999)].

4.1. DEFINITION. *We denote by $\mathcal{D}$ the category of (discrete) R-modules. We define a category $\mathrm{Chu}(\mathcal{D}, R)$ as follows. An object of $\mathrm{Chu}(\mathcal{D}, R)$ is a pair (A, X) in which A and X are R-modules together with a pairing $\langle -,- \rangle : A \otimes X \longrightarrow R$. A morphism $(f, g) : (A, X) \longrightarrow (B, Y)$ consists of R-linear homomorphisms $f : A \longrightarrow B$ and $g : Y \longrightarrow X$ such that $\langle fa, y\rangle = \langle a, gy\rangle$ for $a \in A$ and $y \in Y$. The definition of morphism is equivalent to the commutativity of either of the squares*

$$\begin{array}{ccc} A & \longrightarrow & \hom(X,R) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \hom(g,R)} \\ B & \longrightarrow & \hom(Y,R) \end{array} \qquad\qquad \begin{array}{ccc} Y & \longrightarrow & \hom(B,R) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \hom(f,R)} \\ X & \longrightarrow & \hom(A,R) \end{array} \qquad (*)$$

in which the horizontal arrows are the adjoint transposes of the two $\langle -,- \rangle$.

One way of getting an object of $\mathrm{Chu}(\mathcal{D}, R)$ is to begin with a topological R-module C and forming the pair $(|C|, \hom(C, R))$ with the pairing given by evaluation. Then, if C and C' are topological modules, it follows immediately that every continuous homomorphism from C to C' induces a morphism $(|C|, \hom(C, R)) \longrightarrow (|C'|, \hom(C', R))$.

If $\mathbf{U} = (A, X)$ is an object of $\mathrm{Chu}(\mathcal{D}, R)$, we denote by $\mathbf{U}^{\perp}$ the object (X, A) with the evident pairing. If $\mathbf{U} = (A, X)$ and $\mathbf{V} = (B, Y)$ are objects of $\mathrm{Chu}(\mathcal{D}, R)$, the set of morphisms $\mathbf{U} \longrightarrow \mathbf{V}$ is obviously an R-module that we denote $[\mathbf{U}, \mathbf{V}]$. Then $\mathrm{Chu}(\mathcal{D}, R)$ becomes a $*$-autonomous category when we define

$$\mathbf{U} \multimap \mathbf{V} = ([\mathbf{U}, \mathbf{V}], A \otimes Y)$$

with pairing $\langle (f, g), (a, y)\rangle = \langle fa, y\rangle = \langle a, gy\rangle$. When $\mathbf{R} = (R, R)$ with the R-module structure as pairing, one easily sees that $\mathbf{U} \multimap \mathbf{R} = \mathbf{U}^{\perp}$. We call $\mathbf{R}$ the dualizing object. There is also a tensor product given by

$$\mathbf{U} \otimes \mathbf{V} = (\mathbf{U} \multimap \mathbf{V}^{\perp})^{\perp}$$

We say that the object (A,X) is **separated** if the induced map $A \longrightarrow \hom(X,R)$ is monic and that it is **extensional** if $X \longrightarrow \hom(A,R)$ is monic. (Incidentally, extensionality is the property of functions that two are equal if they are equal for all possible arguments. Thus extensionality here means that, in effect, X is a module of homomorphisms $A \longrightarrow R$). A pair is called **non-singular** if it is both separated and extensional. This means that for all $0 \neq a \in A$ there is an $x \in X$ with $\langle a,x\rangle \neq 0$ and, symmetrically, that for all $0 \neq x \in X$, there is an $a \in A$ with $\langle a,x\rangle \neq 0$. The results in the theorem that follows are proved in detail in [Barr (1998)]. Since $\mathcal{D}$ is abelian the factorization referred to in that citation can only be the standard one into epics and monics. Let $\mathrm{Chu}_s(\mathcal{D},R)$, $\mathrm{Chu}_e(\mathcal{D},R)$, and $\mathrm{chu}(\mathcal{D},R)$ denote, respectively, the full subcategories of $\mathrm{Chu}(\mathcal{D},R)$ consisting of the separated, the extensional, and the separated extensional objects.

4.2. Theorem. [Barr (1998)]

1. *The inclusion* $\mathrm{Chu}_s(\mathcal{D},R) \hookrightarrow \mathrm{Chu}(\mathcal{D},R)$ *has a left adjoint* S*;*

2. *the inclusion* $\mathrm{Chu}_e(\mathcal{D},R) \hookrightarrow \mathrm{Chu}(\mathcal{D},R)$ *has a right adjoint* E*;*

3. $SE \cong ES$*;*

4. *when* (A,X) *is extensional and* (B,Y) *is separated, then* $(A,X) \multimap (B,Y)$ *is separated; equivalently, the tensor product of extensional objects is extensional;*

5. $\mathrm{chu}(\mathcal{D},R)$ *becomes a* $*$*-autonomous category when we define*

$$(A,X) \multimap (B,Y) = E((A,X) \multimap (B,Y))$$

$$(A,X) \otimes (B,Y) = S((A,X) \otimes (B,Y))$$

where the right hand sides of these formulas refer to the operations in $\mathrm{Chu}(\mathcal{D},R)$ *and the left hand sides define the operations in* $\mathrm{chu}(\mathcal{D},R)$.

4.3. Conventions. *Here and later, when* S *is a set and* A *is an object of some category, we denote by* $S \cdot A$ *an* S*-fold coproduct of copies of* A*. When* $A = R$*, this is simply the free* R*-module generated by* S*.*

From now on, we will be restricting our attention to objects of $\mathrm{chu}(\mathcal{D},R)$*, unless explicitly stated otherwise.*

4.4. Self-injective rings. In a number of cases the category of chu objects is equivalent to one or more concrete categories of topological objects. See [Barr & Kleisli (2001)] or the unpublished note [Barr, (unpublished)] for an example that generalizes Pontrjagin duality.

In order to motivate what follows, we begin with the special case in which R is self-injective. This case is treated in more detail in Section 6 below. If $\mathbf{U} = (A,X)$ is an object of $\mathrm{chu}(\mathcal{D},R)$, the separability condition implies that $A \hookrightarrow R^{\|X\|}$. Let $\sigma\mathbf{U}$ denote the topological module whose underlying module is A and whose topology is such that

$\sigma\mathbf{U} \subseteq R^{\|X\|}$ is a topological embedding, with R topologized discretely. We will see in 6.1 below that R continues to be injective in a category $\mathcal{B}$ that contains $\mathcal{C}$ as a full subcategory; *a fortiori* R is injective in $\mathcal{C}$ with respect to topological embeddings.

We claim that $\sigma\mathbf{U}$ determines $\mathbf{U}$. Observe that we can view $\hom(\sigma\mathbf{U}, R)$ as a submodule of $\hom(A, R)$ consisting of the maps continuous in the topology induced by $R^{\|X\|}$. Moreover, each element of X is continuous in that topology, so that we have $X \subseteq \hom(\sigma\mathbf{U}, R) \subseteq \hom(A, R)$. It follows from Corollary 3.3 that the canonical map $\|X\| \cdot R \hookrightarrow \hom(R^{\|X\|}, R)$ is an isomorphism. Injectivity of R applied to the embedding $\sigma\mathbf{U} \longrightarrow R^{\|X\|}$ implies that $\|X\| \cdot R \longrightarrow \hom(\sigma\mathbf{U}, R)$ is surjective. The image of this map is the submodule of $\hom(\sigma\mathbf{U}, R)$ generated by the elements of X, which is X itself, so that $X = \mathrm{Hom}(\sigma\mathbf{U}, A)$. Thus we conclude that $X \cong \hom(\sigma\mathbf{U}, R)$ and so we recover $\mathbf{U}$ from $\sigma\mathbf{U}$ as $(|\sigma\mathbf{U}|, \hom(\sigma\mathbf{U}, R))$.

This argument evidently depends on the injectivity of R. In order to deal with the case that R is not injective (but its complete ring of quotients is) we will have to introduce new subcategories of $\mathrm{chu}(\mathcal{D}, R)$.

4.5. The functors σ and ρ. We introduce functors $\sigma : \mathrm{chu}(\mathcal{D}, R) \longrightarrow \mathcal{C}$ and $\rho : \mathcal{C} \longrightarrow \mathrm{chu}(\mathcal{D}, R)$ as follows. If $\mathbf{U} = (A, X)$ is an object of $\mathrm{chu}(\mathcal{D}, R)$, then $\sigma\mathbf{U}$ is the module A topologized as a subobject of $R^{\|X\|}$. If C is an object of $\mathcal{C}$, let $\rho C = (|C|, |C^\star|)$ with the obvious pairing.

4.6. Proposition. *ρ is left adjoint to σ.*

Proof. As seen in Diagram $(*)$ on Page 4, a map $\rho C \longrightarrow (A, X)$ is given by R-linear homomorphisms $|C| \longrightarrow A$ and $X \longrightarrow |C^\star|$ for which the left-hand square of

$$\begin{array}{ccccc} |C| & \longrightarrow & \mathrm{Hom}(|C^\star|, R) & \longrightarrow & R^{\|C^\star\|} \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{Hom}(X, R) & \longrightarrow & R^{\|X\|} \end{array}$$

commutes, while the right-hand one obviously does. But the commutation of the outer square is the condition required for continuity of $C \longrightarrow \sigma(A, X)$ when A is topologized by the embedding into $R^{\|X\|}$. Thus we get a map $C \longrightarrow \sigma(A, X)$.

In the other direction, a map $C \longrightarrow \sigma(A, X)$ consists of a map $|C| \longrightarrow A$ for which the composite $C \longrightarrow A \longrightarrow R^{\|X\|}$ is continuous. Dualizing gives a map $X \cdot R \longrightarrow |C^\star|$ which when composed with the inclusion $X \longrightarrow X \cdot R$ gives a map $X \longrightarrow |C^\star|$. This function is actually a homomorphism as it factors $X \longrightarrow X \cdot R \longrightarrow \mathrm{Hom}(A, R) \longrightarrow |C^\star|$ and the composite of the first two as well as the third are homomorphisms. The remaining details are left to the reader. ∎

4.7. Proposition. *For any $C \in \mathcal{C}$, the inner adjunction $C \longrightarrow \sigma\rho C$ is an isomorphism.*

Proof. This follows from the facts that $\rho C = (|C|, |C^\star|)$ and that $\sigma\rho C$ is just $|C|$ topologized as a subspace of $R^{\|C^\star\|}$, which is just the original topology on C by 3.1. ∎

Next we identify the objects of $\mathrm{chu}(\mathcal{D},R)$ on which $\rho\sigma$ is the identity. If $\mathbf{U}=(A,X)$ is an object of $\mathrm{chu}(\mathcal{D},R)$, then $\sigma\mathbf{U}=A$, topologized as a subobject of $R^{\|X\|}$. Then $\rho\sigma\mathbf{U}=(A,\hom(\sigma\mathbf{U},R))$. Since the elements of X induce continuous maps on $\sigma\mathbf{U}$, we have $X\longrightarrow\hom(\sigma\mathbf{U},R)\hookrightarrow\hom(A,R)$. Since $\mathbf{U}$ is extensional the composite is monic and hence so is the first map. In other words, we can assume $X\subseteq\hom(\sigma\mathbf{U},R)$. There is, however, no reason to suppose that $X=\hom(\sigma\mathbf{U},R)$ so that $\rho\sigma$ is not generally the identity. To deal with this situation, we introduce new conditions on objects of the chu category.

4.8. Definition. *We say that* $\mathbf{U}$ *is* **high** *when* $\rho\sigma\mathbf{U}=\mathbf{U}$. *We also say that* $\mathbf{U}$ *is* **wide** *when* $\mathbf{U}^{\perp}$ *is* high.

We let $\mathrm{chu}(\mathcal{D},R)_h$, $\mathrm{chu}(\mathcal{D},R)_w$ and $\mathrm{chu}(\mathcal{D},R)_{hw}$ denote the full subcategories of $\mathrm{chu}(\mathcal{D},R)$ consisting, respectively, of the high objects, the wide objects and the high, wide objects.

If (A,X) is an object of $\mathrm{chu}(\mathcal{D},R)$, the topology induced on A by its embedding into R^X has a subbase at 0 given by the kernels of the composites $A\longrightarrow R^X\xrightarrow{p_x}R$. It follows that $\varphi:A\longrightarrow R$ is continuous in this topology if and only if there are finitely many elements $x_1,\ldots,x_n\in X$ such that $\ker\varphi\supseteq\bigcap\ker\langle -,x_i\rangle$. Thus we conclude:

4.9. Proposition. *The object* $(A,X)\in\mathrm{chu}(\mathcal{D},R)$ *is high if and only if, for all* $\varphi:A\longrightarrow R$ *and all* $x_1,\ldots,x_n\in X$ *such that* $\ker\varphi\supseteq\bigcap\ker\langle -,x_i\rangle$, *there is an* $x\in X$ *such that* $\varphi=\langle -,x\rangle$. ∎

4.10. Proposition. *If* $\mathbf{U}=(A,X)\in\mathrm{chu}(\mathcal{D},R)$, *then the cokernel of the map* $\|X\|\cdot R\longrightarrow\sigma\mathbf{U}$ *that is induced by the inclusion* $\mathbf{U}\subseteq R^{\|X\|}$ *is weak torsion.*

Proof. This is immediate from 3.3 and 3.4. ∎

4.11. Proposition. *The inclusion* $\mathrm{chu}(\mathcal{D},R)_h\hookrightarrow\mathrm{chu}(\mathcal{D},R)$ *has a right adjoint* H *and the inclusion* $\mathrm{chu}(\mathcal{D},R)_w\hookrightarrow\mathrm{chu}(\mathcal{D},R)$ *has a left adjoint* W.

Proof. We claim that $H=\rho\sigma$. In fact, suppose $\mathbf{U}$ is high and $\mathbf{V}$ is arbitrary. Then, since σ is full and faithful and ρ is its left adjoint, we have

$$\begin{aligned}\mathrm{Hom}(\mathbf{U},\rho\sigma\mathbf{V})&\cong\mathrm{Hom}(\sigma\mathbf{U},\sigma\rho\sigma\mathbf{V})\cong\mathrm{Hom}(\sigma\mathbf{U},\sigma\mathbf{V})\\&\cong\mathrm{Hom}(\rho\sigma\mathbf{U},\mathbf{V})\cong\mathrm{Hom}(\mathbf{U},\mathbf{V})\end{aligned}$$

which shows the first claim. For the second, let $W\mathbf{V}=(H(\mathbf{V}^{\perp}))^{\perp}$. ∎

4.12. Notation. We will denote $H(A,X)$ by $(A,\bar{X})$ since the first coordinate is A and the second is the subset of $\hom(A,R)$ consisting of the continuous maps. That $X\subseteq\bar{X}$ follows from extensionality. Similarly, we denote $W(A,X)$ by $(\bar{A},X)$. However, since, as we will see in Example 4.19, HW is not naturally equivalent to WH, we will never use the ambiguous $(\bar{A},\bar{X})$ outside of this very sentence.

4.13. Remark. Since $X \hookrightarrow \bar{X}$ is essentially induced from the topological inclusion $A \longrightarrow R^X$, it follows from Theorem 3.4 that the cokernel of $X \hookrightarrow \bar{X}$ is weak torsion.

Recall from 4.1 and Theorem 4.2.4 that when $\mathbf{U} = (A, X)$ and $\mathbf{V} = (B, Y)$ are objects of $\mathrm{chu}(\mathcal{D}, R)$, the tensor product in $\mathrm{Chu}(\mathcal{D}, R)$ is given by $(A, X) \otimes (B, Y) = (A \otimes B, [\mathbf{U}, \mathbf{V}^\perp])$ and is extensional (but not generally separated). Its separated reflection is gotten by factoring out of $A \otimes B$ the elements that are annihilated by every map $\mathbf{U} \longrightarrow \mathbf{V}^\perp$. It nonetheless makes sense to talk of continuous maps $A \otimes B \longrightarrow R$.

In the study of Chu objects that are separated and extensional, a crucial point was that the separated reflection commuted with the extensional coreflection. One would similarly hope here that the wide reflection might commute with the high coreflection. That this fails will be shown in Example 4.19. However, the only real consequence of that hoped-for commutation that matters to us remains true:

4.14. Proposition. *If* $\mathbf{U}$ *is high, so is* $W\mathbf{U}$*; dually if* $\mathbf{U}$ *is wide, so is* $H\mathbf{U}$*.*

Proof. It suffices to prove the first claim. Let $\mathbf{U} = (A, X)$ be high, with $W\mathbf{U} = (\bar{A}, X)$. As noted above in 4.13, the cokernel of $A \hookrightarrow \bar{A}$ is weak torsion. Since weak torsion modules have no non-zero homomorphisms into R, we see that two homomorphisms $\bar{A} \longrightarrow R$ that agree on A are equal. Now suppose that $\varphi : \bar{A} \longrightarrow R$ and $x_1, \ldots, x_n \in X$ such that $\ker\varphi \supseteq \bigcap \ker\langle -, x_i\rangle$. Then $\ker(\varphi|A) \supseteq \bigcap \ker(\langle -, x_i\rangle|A)$. Since A is wide, there is an $x \in X$ such that $\varphi|A = \langle -, x\rangle|A$. But then $\varphi = \langle -, x\rangle$ on all of $\bar{A}$. ∎

4.15. Proposition. *Let* $\mathbf{U} = (A, X)$ *and* $\mathbf{V} = (B, Y)$ *be high and extensional. Then* $\mathbf{U} \otimes \mathbf{V}$ *is high (and extensional).*

Proof. By definition $\mathbf{U} \otimes \mathbf{V} = (A \otimes B, [\mathbf{U}, \mathbf{V}^\perp])$. This means that the topology on $A \otimes B$ is induced by its map to $R^{\mathrm{Hom}(\mathbf{U},\mathbf{V}^\perp)}$. Note that even when $\mathbf{U}$ and $\mathbf{V}$ are separated this map is not necessarily injective. If $\varphi : A \otimes B \longrightarrow R$ is continuous in this topology, then, as in Proposition 4.9, there are maps $(f_1, g_1), \ldots (f_n, g_n) : \mathbf{U} \longrightarrow \mathbf{V}^\perp$ such that $\ker\varphi \supseteq \bigcap \ker(f_i, g_i)$. Here (f_i, g_i) acts on $A \otimes B$ by $(f_i, g_i)(a \otimes b) = \langle b, f_i a\rangle = \langle a, g_i b\rangle$. Now fix $a \in A$. If $b \in \bigcap \ker\langle -, f_i a\rangle$ then $b \in \ker\varphi(a, -)$. Since for any $y \in Y$, the map $\langle -, y\rangle$ is continuous on B, it follows that $\varphi(a, -) : B \longrightarrow R$ is continuous. Since B is high, there is a unique $y = fa \in Y$ such that $\varphi(a \otimes b) = \langle b, fa\rangle$ for all $b \in B$. The usual arguments involving uniqueness show that f is an R-linear homomorphism $A \longrightarrow Y$. There is similarly an R-linear map $g : B \longrightarrow X$ such that $\varphi(a \otimes b) = \langle a, gx\rangle$. It follows that φ is in the image of $[\mathbf{U}, \mathbf{V}^\perp] \longrightarrow \hom(A \otimes B, R)$. Extensionality is proved in [*Barr*(1998)]. ∎

4.16. Corollary. *If* $\mathbf{U}$ *is high and* $\mathbf{V}$ *is wide, then* $\mathbf{U} \multimap \mathbf{V}$ *is wide.*

Proof. This is immediate from the fact that $\mathbf{U} \multimap \mathbf{V} = (\mathbf{U} \otimes \mathbf{V}^\perp)^\perp$. ∎

Now we can define the $*$-autonomous structure on $\mathrm{chu}(\mathcal{D}, R)_{hw}$. Assume that $\mathbf{U}$ and $\mathbf{V}$ are high and wide. Then $\mathbf{U} \otimes \mathbf{V}$ is high and $\mathbf{U} \multimap \mathbf{V}$ is wide. So we define $\mathbf{U} \underset{h}{\multimap} \mathbf{V} = H(\mathbf{U} \multimap \mathbf{V})$ and $\mathbf{U} \underset{w}{\otimes} \mathbf{V} = W(\mathbf{U} \otimes \mathbf{V})$.

4.17. Theorem. *For any* $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W} \in \mathrm{chu}(\mathcal{D}, R)_{hw}$, *we have*

$$\mathrm{Hom}(\mathbf{U} \underset{w}{\otimes} \mathbf{V}, \mathbf{W}) \cong (\mathbf{U}, \mathbf{V} \underset{h}{\multimap} \mathbf{W})$$

Proof.

$$\begin{aligned}\mathrm{Hom}(\mathbf{U} \underset{w}{\otimes} \mathbf{V}, \mathbf{W}) &= \mathrm{Hom}(W(\mathbf{U} \otimes \mathbf{V}), \mathbf{W}) \cong \mathrm{Hom}(\mathbf{U} \otimes \mathbf{V}, \mathbf{W}) \\ &\cong \mathrm{Hom}(\mathbf{U}, \mathbf{V} \multimap \mathbf{W}) \cong \mathrm{Hom}(\mathbf{U}, H(\mathbf{V} \multimap \mathbf{W})) \\ &= \mathrm{Hom}(\mathbf{U}, \mathbf{V} \underset{h}{\multimap} \mathbf{W})\end{aligned}$$

■

Since it is evident that $\mathbf{U} \underset{h}{\multimap} \mathbf{V} \cong \mathbf{V}^{\perp} \underset{h}{\multimap} \mathbf{U}^{\perp}$, we conclude that chu_{hw} is a $*$-autonomous category.

4.18. Theorem. $\mathrm{chu}(\mathcal{D}, R)_{hw}$ *is complete and cocomplete.*

Proof. $\mathrm{chu}(\mathcal{D}, R)_h$ is a coreflective subcategory of $\mathrm{chu}(\mathcal{D}, R)$ and is therefore complete and cocomplete and $\mathrm{chu}(\mathcal{D}, R)_{hw}$ is a reflective subcategory of $\mathrm{chu}(\mathcal{D}, R)_h$ and the same is true of it. ■

4.19. Example. *In general, HW is not naturally isomorphic to WH.*

Assume that R contains an element r that is neither invertible nor a zero divisor. Let $(1/r)R$ denote the R-submodule of the classical ring of quotients (gotten by inverting all the non zero-divisors of R) generated by $1/r$. Then as modules, we have proper inclusions $rR \subseteq R \subseteq (1/r)R$.

Let $\mathbf{U} = (R, R)$ and $\mathbf{V} = (R, rR)$ both using multiplication as pairing. It is clear that $\mathbf{U} \in \mathrm{chu}(\mathcal{D}, R)_{hw}$. As for $\mathbf{V}$, it is evident that $\sigma\mathbf{V} = R$. The topology is discrete since under the inclusion $R \subseteq R^{\|rR\|}$, the only element of R that goes to 0 under the projection on the coordinate r is 0, since r is not a zero-divisor. Then $H\mathbf{V} = \rho\sigma\mathbf{V} = (R, R)$. Moreover if $(f, g) : \mathbf{U} \longrightarrow \mathbf{V}$ is the identity on the first coordinate and inclusion on the second, it is clear that $H(f, g)$ is just the identity. Since $H\mathbf{V}$ is obviously also wide, we see that $WH(f, g)$ is also the identity.

Now let us calculate $HW\mathbf{V}$. Since $\mathbf{V}^{\perp} = (rR, R)$, it is clear that $\sigma(\mathbf{V}^{\perp}) = rR$. Moreover $\sigma(g, f) : \sigma(\mathbf{V}^{\perp}) \longrightarrow \sigma(\mathbf{U}^{\perp})$ is the proper inclusion $rR \subseteq R$ and is not the identity. We can identify $\rho\sigma(\mathbf{V}^{\perp})$ as $(rR, (1/r)R)$ so that $W\mathbf{V} = ((1/r)R, rR)$, which is isomorphic to $\mathbf{U}$ and is therefore high and wide. Thus $HW\mathbf{V} = W\mathbf{V}$ (assuming, as we may, that H is the identity on chu_h and W is the identity on chu_w). So $WH(f, g)$ is an isomorphism but $HW(f, g)$ is not, which readily implies that WH cannot be naturally isomorphic to HW. ■

5. The $*$-autonomous structure in $\mathcal{C}$

Recall that for $C \in \mathcal{C}$, $C^{\star}$ is the object $\mathrm{hom}(C, R)$ topologized as a subset of $R^{\|C\|}$.

5.1. Proposition. *The map $C \longrightarrow C^{\star\star}$ that takes an element of C to evaluation at that element is a topological embedding.*

Proof. By hypothesis, there is an embedding $C \hookrightarrow R^S$ for some set S. This transposes to a function $S \longrightarrow \mathrm{Hom}(C, R) = \|C^\star\|$ and then $R^{\|C^\star\|} \longrightarrow R^S$. The diagram

$$\begin{array}{ccc} C & \hookrightarrow & R^S \\ \downarrow & & \uparrow \\ C^{\star\star} & \hookrightarrow & R^{\|C^\star\|} \end{array}$$

commutes and the result is that the left-hand arrow, being an initial factor of an embedding, is one itself. ∎

We say that the object C is **reflexive** if the canonical map $C \longrightarrow C^{\star\star}$ is an isomorphism. Let $\mathcal{C}_r$ denote the full subcategory of reflexive objects.

5.2. Proposition. *For any $C \in \mathcal{C}$, the object $C^\star$ is reflexive.*

Proof. We have $C \hookrightarrow C^{\star\star}$ which gives $C^{\star\star\star} \longrightarrow C^\star$ and composes with the canonical $C^\star \longrightarrow C^{\star\star\star}$ to give the identity. Thus $C^{\star\star\star} \cong C^\star \oplus C'$ for a submodule $C' \subseteq C^{\star\star\star}$. In particular, C' is weak torsion free. On the other hand, the inclusion $C^{\star\star} \longrightarrow R^{\|C^\star\|}$ gives a map $\|C^\star\| \cdot R \longrightarrow C^{\star\star\star}$ whose cokernel T is, by 3.4, weak torsion. But since the canonical map $\|C^\star\| \cdot R \longrightarrow C^\star$ is obviously surjective, we conclude that $T \cong C'$, which implies that both are zero. ∎

5.3. Proposition. *Let $\mathbf{U} \in \mathrm{chu}(\mathcal{D}, R)$ be high. Then $\sigma\mathbf{U}$ is reflexive if and only if $\mathbf{U}$ is also wide.*

Proof. Let $\mathbf{U} = (A, X)$. In general $(\sigma\mathbf{U})^\star = (\bar{X}, A)$, which is the same as $(X, A) = \sigma(\mathbf{U}^\perp)$ since $\mathbf{U}$ is assumed high. The same argument implies that $(\sigma\mathbf{U})^{\star\star} = \sigma(\bar{A}, X)$ and this is $\sigma\mathbf{U}$ if and only if $\bar{A} = A$, that if and only if is $\mathbf{U}$ is also wide. ∎

5.4. Corollary. *The functors σ and ρ induce inverse equivalences between the categories $\mathrm{chu}(\mathcal{D}, R)_{hw}$ and $\mathcal{C}_r$.* ∎

5.5. The internal hom in $\mathcal{C}_r$. The obvious candidate for an internal hom in $\mathcal{C}_r$ is to let $C \multimap C'$ be $\mathrm{hom}(C, C')$ embedded as a topological subobject of $R^{\|C\| \times \|C'\|}$. But it is not obvious that this defines a reflexive object. With the help of the $*$-autonomous structure on $\mathrm{chu}(\mathcal{D}, R)_{hw}$, we can prove this.

5.6. Theorem. *For C, $C' \in \mathcal{C}_r$, the object $C \multimap C'$, as defined in the preceding paragraph, is reflexive. In fact, $C \multimap C' \cong \sigma(\rho C \multimap_h \rho C')$.*

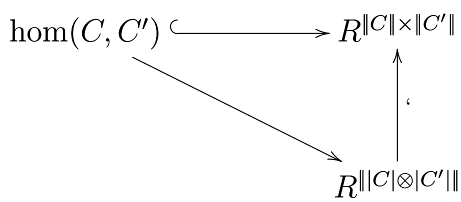
Proof. For this argument, when $C, C' \in \mathcal{C}_r$, we let $C \multimap_r C' = \sigma(\rho C \multimap_h \rho C')$. We wish to show that $C \multimap_r C' = C \multimap C'$. We have that

$$\begin{aligned} C \multimap_r C' &= \sigma(\rho C \multimap_h \rho C') = \sigma H(\rho C \multimap \rho C') \\ &= \sigma\rho\sigma(\rho C \multimap \rho C') && \text{(because } H = \rho\sigma\text{, 4.11)} \\ &= \sigma(\rho C \multimap \rho C') && \text{(from 4.7)} \end{aligned}$$

Now, we have

$$\rho C \multimap \rho C' = ([\rho C, \rho C'], |C| \otimes |C'^\star|) \cong (\hom(C, C'), |C| \otimes |C'^\star|)$$

since ρ is full and faithful on $\mathcal{C}_r$. Thus $\sigma(\rho C \multimap \rho C')$ is just $\hom(C, C')$ equipped with the topology it inherits from $R^{\||C|\otimes|C'^\star|\|}$. But the definition of $C \multimap C'$ is just $\hom(C, C')$ equipped with the topology it inherits from $R^{\|C\|\times\|C'^\star\|}$ so it suffices to show that those topologies coincide. There is a natural map $\|C\| \times \|C'^\star\| \longrightarrow \||C| \otimes |C'^\star|\|$ which gives rise to a map $R^{\|C\otimes C'^\star\|}$. From the commutative triangle



we see that $\hom(C, C') \longrightarrow R^{\||C|\otimes|C'|\|}$, as a first factor in a topological embedding, is also a topological embedding. Thus $C \multimap C'$ is just $\sigma(\rho C \multimap_h \rho C')$. ∎

6. The case of a self-injective ring

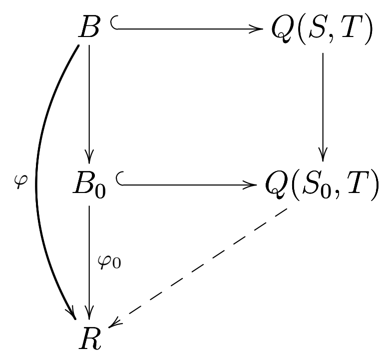
Things get simpler and a new possibility opens when R is a commutative self-injective ring. This was done in [Barr (1999)] when R is a field and what was done there goes through unchanged when R is separable (that is, a product of finitely many fields), but there are many more rings that are self-injective, for example, an arbitrary product of fields, and any complete boolean ring.

Throughout this section we will suppose that R is a commutative self-injective ring.

Let S and T be sets. We denote by $Q(S, T)$ the module $R^S \times |R^T|$. The module R itself is always topologized discretely. We let $\mathcal{B}$ denote the category of topological R-modules that can be embedded into a module of the form $Q(S, T)$ for some sets S and T. The principal use we make of self-injectivity is contained in the following.

6.1. Theorem. *R is injective in $\mathcal{B}$ with respect to topological inclusions.*

Proof. (Compare the proof of Theorem 3.2) It is easy to reduce this to the case of an object embedded into an object of the form $Q(S,T)$. So we begin with $B \hookrightarrow Q(S,T)$ and a continuous homomorphism $\varphi : B \longrightarrow R$. The topology on $Q(S,T)$ has a basis at 0 of sets of the form $Q(S - S_0, T)$ for a finite subset $S_0 \subseteq S$. Then the kernel of φ contains a set of the form $B \cap Q(S - S_0, T)$. Let $B_0 = B/(B \cap Q(S - S_0, T))$. Then we get a commutative diagram



in which the diagonal map exists because B_0 and $Q(S_0,T)$ are discrete and R is injective. ■

6.2. Corollary. *When R is injective, every object of* chu$(\mathcal{D}, R)$ *is high and wide.*

Proof. The proof is left as a simple exercise (see the discussion in 4.4). ■

We will say that an object of $\mathcal{B}$ is **weakly topologized** if it is embedded in a power of R (with the product topology). Let $\mathcal{B}_{\rm wk}$ be the full subcategory of $\mathcal{B}$ consisting of the weakly topologized objects. Evidently $\mathcal{B}_{\rm wk}$ is the same as $\mathcal{C}$, as used before this section. It is clear that every object of chu$(\mathcal{D}, R)$ is both high and wide and thus σ and ρ, as defined in 4.5 are actually equivalences between $\mathcal{B}_{\rm wk}$ and chu$(\mathcal{D}, R)$. Since chu$(\mathcal{D}, R)$ is $*$-autonomous, so is $\mathcal{B}_{\rm wk}$. Thus we have,

6.3. Theorem. *The category $\mathcal{B}_{\rm wk}$ of R-cogenerated topological algebras is $*$-autonomous when $B \multimap B'$ is defined as* hom(B, B') *topologized as subspace of $B'^{\|B\|}$.* ■

6.4. The strongly topologized objects. Let $B \in \mathcal{B}$. We say that B is **strongly topologized** if whenever $B' \longrightarrow B$ is a bijection in $\mathcal{B}$ that induces an isomorphism $\mathrm{Hom}(B, R) \longrightarrow \mathrm{Hom}(B', R)$, then $B' \longrightarrow B$ is an isomorphism. We have already denoted the category of weakly topologized modules by $\mathcal{B}_{\rm wk}$ and we denote by $\mathcal{B}_{\rm st}$ the full subcategory of strongly topologized modules.

For $B \in \mathcal{B}$, let $S(B)$ index the set of all topological R-modules that have the same underlying R-module as B, a stronger topology and the same set of continuous homomorphisms into R.

6.5. Theorem. *Let B be an object of $\mathcal{B}$. Then among all objects B_s with $s \in S(B)$, there is one whose topology is finer than all the others.*

Proof. Let τB be defined so that

$$\begin{array}{ccc} \tau B & \longrightarrow & \prod_{s\in S} B_s \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow[\text{diag}]{} & B^S \end{array}$$

is a pullback. Since $\prod B_s \longrightarrow B^S$ is a bijection, so is $\tau B \longrightarrow B$ so it suffices to show that $\mathrm{Hom}(B,R) \longrightarrow \mathrm{Hom}(\tau B,R)$ is an isomorphism. It is clearly monic, so it suffices to see that it is surjective. In this diagram, the bottom arrow is a topological embedding, hence so is the top arrow. If we apply the functor $\hom(-,R)$ to the above diagram and use Corollary 3.3, we get

$$\begin{array}{ccc} \hom(\tau B,R) & \twoheadleftarrow & \sum \hom_{s\in S}(B_s,R) \\ \big\uparrow & & \big\uparrow \cong \\ \hom(B,R) & \longleftarrow & S\cdot \hom(B,R) \end{array}$$

from which we see immediately that $\hom(B,R) \longrightarrow \hom(\tau B,R)$ is surjective. ∎

This theorem and its proof are essentially identical to that of [Barr & Kleisli (2001), Theorem 4.1 (proof that 2 implies 3)]. Another approach is used for the similar [Barr & Kleisli (1999), Proposition 3.8]. The latter paper omitted, however to prove that τ was functorial. We give the proof here, essentialy the same as that of the former citation.

6.6. Theorem. *The object function τ determines a functor $\tau : \mathcal{B} \longrightarrow \mathcal{B}_{\mathrm{st}}$ which is left adjoint to the inclusion.*

Proof. It will be sufficient to show that whenever $B' \longrightarrow B$ is an arrow in $\mathcal{B}$ and $B_s \longrightarrow B$ for $s \in S(\mathcal{B})$, then there is an $s' \in S(B')$ and a commutative square

$$\begin{array}{ccc} B_{s'} & \longrightarrow & B' \\ \big\downarrow & & \big\downarrow \\ B_s & \longrightarrow & B \end{array}$$

We let B'' be a pullback $B_s \times_B B'$. Since B_s has the same underlying module as B, we can assume that B'' has the same underlying module as B'. Thus it suffices to show it has the same set of continuous maps to R as B' does. The map $B'' \longrightarrow B_s \times B'$ is an equalizer of two maps $B_s \times B' \longrightarrow B$ so that B'' is embedded in the product $B_s \times B$. Then we have a commutative square

$$\begin{array}{ccc} B'' & \hookrightarrow & B' \times B_s \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B' \times B \end{array}$$

If we apply the functor $\hom(-, R)$ we get the square

$$\begin{array}{ccc} \hom(B'', R) & \twoheadleftarrow & \hom(B', R) \times \hom(B_s, R) \\ \uparrow & & \uparrow \cong \\ \hom(B', R) & \longleftarrow & \hom(B', R) \times \hom(B, R) \end{array}$$

from which it follows that $\hom(B', R) \longrightarrow \hom(B'', R)$ is surjective and it is obviously injective so that $B'' = B_{s'}$ for some $s' \in S(B')$. This shows that τ is a functor and the adjunction is clear. ∎

If we let ωB denote the module $|B|$ retopologized by its embedding into $R^{\mathrm{Hom}(B,R)}$ then ωB is the weakest topology with the same set of homomorphisms into R as B and τB is the strongest. We leave it to the reader to prove the easy fact that ω is right to the inclusion of $\mathcal{B}_{\mathrm{wk}}$ into $\mathcal{B}$. Clearly $\tau\omega = \tau$ and $\omega\tau = \omega$.

6.7. Theorem. *τ and ω induce inverse equivalences between $\mathcal{B}_{\mathrm{wk}}$ and $\mathcal{B}_{\mathrm{st}}$.*

Proof. If $B \in \mathcal{B}_{\mathrm{wk}}$, then $\tau B \in \mathcal{B}_{\mathrm{st}}$ and $\omega\tau B = \omega B = B$, while if $B \in \mathcal{B}_{\mathrm{st}}$, then $\omega B \in \mathcal{B}_{\mathrm{wk}}$ and $\tau\omega B = \tau B = B$. ∎

Consequently, $\mathcal{B}_{\mathrm{st}}$ is also $*$-autonomous. The internal hom is gotten by first forming $B \multimap B'$ and then applying τ.

7. Discussion

It is obvious that the $*$-autonomous structure on the category $\mathcal{C}_r$ depends crucially on 5.6. But we could not find a proof of that fact independent of the chu category, in particular, the high wide subcategory. While it is certainly true that you use the methods that work, an independent argument would still be desirable.

For example, suppose R is a ring that is not necessarily commutative. The category of two-sided R-modules has an obvious structure of a biclosed monoidal category. If A is a topological module, it has both a left dual ${}^*A$ (consisting of the left R-linear homomorphisms into R) and a right dual A^*. The two duals commute and there is a canonical map $A \longrightarrow {}^*A^*$. It is natural to call an object reflexive if that map is an isomorphism. One can now ask if the internal homs (that is, the left and right homs) of two reflexive objects is reflexive. It might require something like the left and right complete rings of quotients being isomorphic and also R-injective. Even the case that R has no zero divisors would be interesting.

Although there a Chu construction for a biclosed monoidal category, with an infinite string of left and right duals (see [Barr, 1995]), there does not seem to be any obvious way of defining separated or extensional objects. An object might be separated, say, with respect to its right dual, but not its left. Factoring out elements that are annihilated by the left dual would usually lead to the right dual being undefined. And even if a notion of separated, extensional objects was definable, what possible functor to the topological category would exist that would transform to all the infinite string of duals? Thus we would require an independent argument for the analog of Theorem 5.6.

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