ON SUBGROUPS OF THE LAMBEK PREGROUP

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ABSTRACT. A pregroup is a partially ordered monoid in which every element has a left and a right adjoint. The main result is that for some well-behaved subgroups of the group of diffeomorphisms of the real numbers, the set of all endofunctions of the integers that are asymptotic at $\pm\infty$ to (the restriction to the integers of) a function in the subgroup is a pregroup.

1. Introduction

In [Lambek, to appear], Lambek has described a new kind of not quite algebraic structure. A **pregroup** is a partially ordered monoid in which multiplication by every element has a left and right adjoint. More precisely, a poset P that is also a monoid and such that for each element $x \in P$ there are elements x^ℓ, $x^r \in P$ such that

$$x^\ell x \leq 1 \leq xx^\ell \qquad \text{and} \qquad xx^r \leq x^r x$$

It is easy to see that the underlying functor from pregroups to posets has a left adjoint and is, in fact, tripleable. Thus there is one ready source of pregroups: the free ones generated by a poset.

In addition to the free pregroups (and of course, the groups with the discrete partial order), there are very few known examples. The best known is what we will call the **Lambek pregroup** after its discoverer. This is the set of all functions $\mathbf{Z} \to \mathbf{Z}$ that are unbounded in both directions, under composition. We will denote this pregroup by L.

If you try to analyze why this works in order to find other examples of the same sort, you appear to hit a blank wall. The problem is not only showing that every function has a left and right adjoint, but also showing that the adjoints have the same properties. Having a right and left adjoint means preserving all infs and sups. In $\mathbf{Z}$ only finite non-empty sets have infs and sups and they are in the set and so preserved by every functor. Only a total order in which every bounded set is finite can have that property. For a bounded function (in either direction) some of the infs and sups required for the adjoint will be empty and that does not work. If you had a first (or last element) then it would have to be preserved and there is no way of forcing the adjoint to preserve them. So you need a total order without first or last element in which every bounded set is finite. These properties are easily seen to characterize $\mathbf{Z}$ as an ordered set.

Since there is no obvious way to generalize the construction of L, we turn instead to finding sub-pregroups of L. One source of functions in L is to take an increasing bijection $\phi : \mathbf{R} \to \mathbf{R}$ and define $f(n) = \lfloor \phi(n) \rfloor$ (this is the floor, or greatest integer, function).

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In fact, we will see that every element of L has that form and we can even take ϕ to be differentiable. Thus one way to explore sub-pregroups of L is to look at subgroups of the group G of increasing diffeomorphisms of $\mathbf{R}$. Not every subgroup will do; it is necessary to impose a condition that the functions grow at a moderate rate. Exponentials grow too fast and logarithms too slowly, but, for example, polynomials are, like mama bears, just right.

In this paper, we will be comparing the behavior of functions at $\pm\infty$. In particular, we will write $\phi \sim \psi$ if

$$\lim_{x \to \pm\infty} \frac{\phi(x)}{\psi(x)} = 1$$

For the most part, we will study this only at $+\infty$, since the transformation $\phi \mapsto \phi^*$ defined by $\phi^*(x) = -\phi(-x)$ changes the behavior at $-\infty$ to that at $+\infty$ [1].

2. Preliminary results

In this paper, we will be dealing with functions $\mathbf{Z} \to \mathbf{Z}$, which we will denote by small Roman letters, such as f and functions $\mathbf{R} \to \mathbf{R}$ that we will denote by small Greek letters, such as ϕ. Unless explicitly mentioned otherwise, the graphs of the real valued functions will be assumed to have tangents everywhere, although they might be vertical. As an example of the latter, for any positive real a, we let $()^{(a)}$ denote the **symmetric ath power function** defined by

$$x^{(a)} = \begin{cases} x^a & \text{if } x \geq 0 \\ -(-x)^a & \text{if } x < 0 \end{cases}$$

which has a vertical tangent at 0 when $a < 1$.

If $x \in \mathbf{R}$, we define, as usual, the floor of x, denoted $\lfloor x \rfloor$ to be the largest integer not larger than x and the ceiling of x, denoted $\lceil x \rceil$ as the smallest integer not smaller than x. If $\phi : \mathbf{R} \to \mathbf{R}$ is a function, we let $\lfloor \phi \rfloor : \mathbf{Z} \to \mathbf{Z}$ denote the function such that $\lfloor \phi \rfloor (n) = \lfloor \phi(n) \rfloor$. We define $\lceil \phi \rceil$ similarly.

We let G denote the group of functions $\mathbf{R} \to \mathbf{R}$ that are increasing homomorphisms that have a tangent everywhere (possibly vertical) and whose inverse does too. If ϕ, $\psi \in G$, we will say that $\phi \sim \psi$ if $\lim_{x\to+\infty} \phi(x)/\psi(x) = \lim_{x\to-\infty} \phi(x)/\phi(x) = 1$.

Let L denote the Lambek pregroup of unbounded increasing functions $\mathbf{Z} \to \mathbf{Z}$. If f, $g \in L$, we say that $f \sim g$ if $\lim_{n\to+\infty} f(n)/g(n) = \lim_{n\to-\infty} f(n)/g(n) = 1$. We will also say that if $f \in L$ and $\phi \in G$, then $f \sim \phi$ if $f \sim \lfloor \phi \rfloor$. Since $\phi(x) \to \infty$ as x does, it is obvious that this is the same as $f \sim \lceil \phi \rceil$.

2.1. PROPOSITION. *Every $f \in P$ has the form $\lfloor \phi \rfloor$ for some $\phi \in G$.*

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[1] I would like to thank the referee for suggesting several improvements in the paper and for saving from making an embarrasing error in the proof of 3.

Proof. If n is an integer, suppose that k is the least integer such that $f(n) < f(n+k)$. If $k = 1$, then for $x \in [n, n+1]$, define $\psi(x) = (f(n)+.3)(n+1-x)+(f(n+1)+.3)(x-n)$. Otherwise, for $x \in [n, n+k]$, define

$$\psi(x) = \frac{(f(n)(n+k-1-x)+.3)+(f(n)+.7)(x-n)}{k-1}$$

for $x \le k-1$ and,

$$\psi(x) = (f(n)+.7)(n+k-x)+(f(n+k)+.3)(x-n-k+1)$$

for $x \ge k-1$. What this function does is interpolate linearly between the points $(n, f(n))$ and $(n+k-1, f(n)+.7)$ and then between the latter point and $(n+k, f(n+k))$.

To this point we have constructed a continuous, but not differentiable, function whose integer part at integers is f. It has the additional property that at integers its value differs from the nearest integer by at least .3 and that it is linear between any integer and the next. Now suppose n is an integer and that the slope of ψ before n is m_1 and after n is m_2. Since m_1 and m_2 are positive, there is for any $\epsilon > 0$ a smooth spline function in the interval $[n-\epsilon, n+\epsilon]$ whose values and slopes at $n-\epsilon$ and $n+\epsilon$ are the same as those of ψ and whose slope in the entire interval is between m_1 and m_2. It follows that such a function remains monotonically increasing. Moreover, if ϵ is chosen sufficiently small, the value at n will differ from $\psi(n)$ by less than .3. Thus the function ϕ can be gotten by replacing all the corners by appropriately chosen splines. For technical reasons, we will always take $\epsilon \le .1$. ∎

3. Modest functions

We say that a function $\phi \in G$ is **modest** if whenever ψ, $\rho \in G$ are such that $\psi \sim \rho$, then $\phi\psi \sim \phi\rho$.

3.1. Proposition. *If ϕ, $\theta \in G$ are modest, so is $\phi\theta$.* ∎

I would like to thank G.A. Edgar for the elegant proof of the following theorem that replaces my original, much longer, argument.

3.2. Theorem. *If $\phi \in G$ is a function for which $x\phi'(x)/\phi(x)$ is bounded outside a bounded set, then ϕ is modest.*

Proof. We consider the situation only at $+\infty$; the transformation $\phi \mapsto \phi^*$ (which leaves $x\phi'(x)/\phi(x)$ invariant) takes care of the case at $-\infty$. Since all the functions considered go to ∞ as x does, they are eventually positive. Thus, given ψ and ρ, there is some bounded interval outside of which $x\phi'(x)/\phi(x)$ is bounded and ψ and ρ are positive. Let $\sigma(x) = \log(\psi(x))$ and $\tau(x) = \log(\rho(x))$. Then $\lim_{x\to\infty} \sigma(x) - \tau(x) = 0$. If we define $\hat{\phi}$ by $\hat{\phi}(x) = \log(\phi(\exp(x)))$, we have that

$$\hat{\phi}'(x) = \frac{\exp(x)\phi'(\exp(x))}{\phi(\exp(x))}$$

which, if we let $y = \exp(x)$, gives

$$\hat{\phi}'(x) = \frac{y\phi'(y)}{\phi(y)}$$

so that our hypothesis implies that $\hat{\phi}'(x)$ is bounded. But there is some u between $\sigma(x)$ and $\tau(x)$ for which

$$\hat{\phi}(\sigma(x)) - \hat{\phi}(\tau(x)) = \hat{\phi}'(u)(\sigma(x) - \tau(x))$$

Since $\sigma(x)$ and $\tau(x)$ go to infinity when x does, we are eventually in the range where $\phi'(u)$ is bounded and then it follows that

$$\lim_{x\to\infty} \hat{\phi}(\sigma(x)) - \hat{\phi}(\tau(x)) = 0$$

Take exponentials and replace σ and τ by their definitions to get,

$$\lim_{x\to\infty} \frac{\phi(\psi(x))}{\phi(\rho(x))} = 1$$

■

3.3. PROPOSITION. *Suppose $\psi = \phi^{-1}$. Then for $u = \psi(x)$, we have*

$$\frac{x\psi'(x)}{\psi(x)} = \left(\frac{u\phi'(u)}{\phi(u)}\right)^{-1}$$

PROOF. By differentiating $\phi(\psi(x)) = x$, we get that $\phi'(\psi(x))\psi'(x) = 1$ or $\psi'(x) = 1/\phi'(\psi(x))$. This gives

$$\frac{\psi'(x)x}{\psi(x)} = \frac{x}{\psi(x)\phi'(\psi(x))} = \frac{\phi(\psi(x))}{\psi(x)\phi'(\psi(x))}$$

If we now let $u = \psi(x)$, this becomes

$$\frac{x\psi'(x)}{\psi(x)} = \frac{\phi(u)}{u\phi'(u)}$$

■

3.4. COROLLARY. *For $\phi \in G$, a sufficient condition that both ϕ and ϕ^{-1} be modest is that $x\phi'(x)/\phi(x)$ be both bounded and bounded away from 0 outside a bounded set.*

3.5. PROPOSITION. *Suppose ϕ, $\psi \in G$ are such that $\phi \sim \psi$. If ϕ^{-1} (or ψ^{-1}) is modest, then $\phi^{-1} \sim \psi^{-1}$.*

PROOF. Since $\psi(x)$ goes to ∞ when x does, it is sufficient to show that

$$\lim_{x\to\infty} \frac{\phi^{-1}(\psi(x))}{\psi^{-1}(\psi(x))} = 1$$

This is the same as

$$\lim_{x\to\infty} \frac{\phi^{-1}(\psi(x))}{\phi^{-1}(\phi(x))} = 1$$

and the conclusion follows from the modesty of ϕ^{-1}. ■

Some hypothesis on the growth of ϕ^{-1} is required. If $\phi(x)$ is any function in G that is $\log x$ for $x \geq e$ and $-\log(-x)$ for $x \leq -e$, while $\psi(x) = 1 + \phi(x)$, then $\phi \sim \psi$, while $\phi^{-1} \not\sim \psi^{-1}$.

4. The main theorem

4.1. Proposition. *Suppose that $f \in L$ and ϕ, $\psi \in G$. If ϕ and ψ are modest, $f = \lfloor \phi \rfloor$, and $f \sim \psi$ then $\phi \sim \psi$.*

Proof. The conditions imply that for $n = \lfloor x \rfloor$

$$\frac{\psi(x)}{\phi(x)} = \frac{\psi(x)}{\psi(n)} \frac{\psi(n)}{f(n)} \frac{f(n)}{\phi(n)} \frac{\phi(n)}{\phi(x)}$$

all four factors of which go to 1 as $x \to \infty$. ∎

4.2. Proposition. *Suppose that $f \in L$ and $\phi \in G$ such that $f \sim \lfloor \phi \rfloor$. If ϕ^{-1} is modest, then $f^\ell \sim \phi^{-1} \sim f^r$.*

Proof. Let $\psi \in G$ be such that $f = \lfloor \psi \rfloor$. By perturbing it slightly, we can suppose that $\psi(n)$ is never an integer when n is. This point is not crucial, but it will simplify things. We have $\psi \sim f \sim \phi$. Then $f^\ell(n) \leq m$ if and only if $n \leq f(m)$ if and only if $n \leq \lfloor \psi(m) \rfloor$ if and only if $n \leq \psi(m)$ if and only if $\psi^{-1}(n) \leq m$ if and only if $\lceil \psi^{-1}(n) \rceil \leq m$. Thus $f^\ell = \lceil \psi^{-1} \rceil$. It follows from Proposition 3 that $f^\ell = \lceil \psi^{-1} \rceil \sim \psi^{-1} \sim \phi^{-1}$. As for f^r, just modify our previous constructions to choose ψ so that $f = \lceil \psi \rceil$ and reverse all the inequalities. ∎

4.3. Proposition. *Suppose that f, $g \in L$ and ϕ, $\psi \in G$ are such that $f \sim \phi$ and $g \sim \psi$. If ϕ is modest, then $fg \sim \phi\psi$.*

Proof. We have

$$\frac{f(g(n))}{\phi(\psi(n))} = \frac{f(g(n))}{\phi(g(n))} \frac{\phi(g(n))}{\phi(\psi(n))}$$

The first term goes to 1 as $n \to \infty$ since ψ grows to infinity. As for the second, just use the same argument as in the proof of Theorem 3: Since $\lim \log g(n) - \log \psi(n) = 0$, it follows that $\lim \hat{\phi}(\log g(n)) - \lim \hat{\phi}(\log \psi(n)) = 0$ and then that

$$\lim \frac{\phi(g(n))}{\phi(\psi(n))} = 1$$ ∎

By iterating this, we see that all of $f^{\ell\ell}$, $f^{\ell\ell\ell\ell}$, $\ldots$, f^{rr}, f^{rrrr} are asymptotic to ϕ and all the odd adjoints are asymptotic to ϕ^{-1}. Thus we have, 4.4. Theorem. *Suppose that H is a subgroup of G such that every element of H is modest. Then the subset of L consisting of those functions that are asymptotic to an element of H are a subpregroup of L.*

If the subgroup H of G is generated by functions ϕ all of which satisfy that $x\phi'(x)/\phi(x)$ are both bounded and bounded away from 0, then every element of H is modest. For example, the subgroup generated by a symmetric ath power has that property. If we take a set A of such symmetric power functions, the subgroup of G they generate is all the symmetric power functions for which the exponent lies in the multiplicative subgroup of positive reals generated by A (which is dense unless the subgroup is actually cyclic). One can modify this example by taking x^a for $x \geq 0$ and $-(-x)^b$ when $x < 0$. This will fail to be differentiable at 0 if a and b are different sides of 1, but one can readily smooth that out without changing the behaviour at infinity.

For another example, take the function that for positive x is $x^a \log x$ and the symmetric function for negative x. Examples of this sort are easily created and we will refrain from describing more.

5. Modesty in L

For $f \in L$, let $f'(n) = f(n+1) - f(n)$.

5.1. Theorem. *Suppose that $f \in L$ is such that $|nf'(n)/f(n)|$ is bounded. Then there is a modest function $\phi \in G$ such that $f = \lfloor \phi \rfloor$.*

Proof. We carry out the argument at $+\infty$; the argument at $-\infty$ is similar. Let B be a bound on $|nf'(n)/f(n)|$. Begin with a piecewise linear automorphism ψ such that $f = \lfloor \psi \rfloor$. We can suppose that ψ is chosen so that whenever $f(n) = f(n+1)$, the slope of ψ in the interval $(n, n+1)$ is at most $Bf(n)/(n+1)$ and when $f(n) < f(n+1)$, then $\psi'(x) < 2f'(n)$ in that interval. Then ϕ is constructed from ψ by using Bezier splines at the corners to smooth things. It is clear that for any $x \in [n, n+1]$, the slope of ϕ will be between that of ψ in the interval in question and the two adjacent ones. Thus the slope of ϕ at a point $x \in [n, n+1]$ is does not exceed the maximum of the following quantities:

$$\frac{Bf(n-1)}{n}, \frac{Bf(n)}{n+1}, \frac{Bf(n+1)}{n+2}, 2f'(n-1), 2f'(n), 2f'(n+1)$$

We claim that each of these is less than $4Bf(n)/(n+1)$ for all sufficiently large n. For the first three, it is sufficient to show that $f(n+1)/n < 4f(n)/(n+1)$ for sufficiently large n. We have that

$$\frac{f(n+1)}{f(n)} - 1 = \frac{f(n+1) - f(n)}{f(n)} = \frac{f'(n)}{f(n)} \leq \frac{B}{n}$$

which implies that

$$\frac{f(n+1)}{f(n)} \leq 1 + \frac{B}{n} \qquad \text{or} \qquad f(n+1) \leq \left(1 + \frac{B}{n}\right) f(n)$$

This gives that

$$\frac{f(n+1)}{n} \leq \left(1 + \frac{B}{n}\right) \frac{n+1}{n} \frac{f(n)}{n+1}$$

and it is clear that

$$\left(1 + \frac{B}{n}\right) \frac{n+1}{n} < 4$$

for sufficiently large n. The other three quantities have to be handled individually because f' is not necessarily monotone. From

$$\frac{(n+1)f'(n-1)}{f(n)} \leq \frac{n+1}{n-1} \frac{(n-1)f'(n-1)}{f(n-1)} < 2B$$

for $n \geq 3$, it follows that $2f'(n-1) \leq 4Bf(n)/(n+1)$. For the second term it suffices to observe that $n+1 \leq 2n$ for all $n \geq 1$. For the third, it follows from

$$\frac{(n+1)f'(n+1)}{f(n)} = \frac{(n+1)f'(n+1)}{f(n+1)} \frac{f(n+1)}{f(n)} \leq B\left(1 + \frac{B}{n}\right) \leq 2B$$

for sufficiently large n. This demonstrates that

$$\phi'(x) \leq \frac{4Bf(n)}{n+1} < \frac{4B\phi(x)}{x}$$

which implies that

$$\frac{x\phi'(x)}{\phi(x)} \leq 4B$$

■

References

[Lambek, to appear] Lambek, J. to appear. What are pregroups? This volume.

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