SEARCHING FOR MORE ABSOLUTE $\mathcal{CR}$-EPIC SPACES

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ABSTRACT. We continue our examination of absolute $\mathcal{CR}$-epic spaces, or spaces with the property that any embedding induces an epimorphism, in the category of commutative, rings between their rings of continuous functions. We examine more closely the deleted plank construction, which generalizes the Dieudonné construction, and yields absolute $\mathcal{CR}$-epic spaces which are not Lindelöf. For the Lindelöf case, an earlier paper has shown the usefulness of the countable neighbourhood property, CNP, and the Alster condition (where CNP means that the space is a P-space in any compactification and the Alster condition says that any cover of the space by G_δ spaces has a countable subcover, provided each compact subset can be covered by a finite subset.) In this paper, we find further properties of Lindelöf CNP spaces and of Alster spaces, including constructions that preserve these properties and conditions equivalent to these properties. We explore the outgrowths of such spaces and find several examples that answer open questions in our previous work.

1. Introduction

Unless explicitly stated otherwise, all spaces in this paper are assumed to be Tychonoff (completely regular Hausdorff) and all functions are assumed continuous. We denote by $C(X)$ the commutative unitary ring of real-valued continuous maps on X, with pointwise addition and multiplication and by $C^*(X)$ the subring of $C(X)$ consisting of the bounded functions.

There is a functor β that is left adjoint to the inclusion of the category of Tychonoff spaces into all spaces. For any space X the inner adjunction $X \longrightarrow \beta X$ is a topological embedding if and only if X is Tychonoff. Since the unit interval $[0,1]$ is compact, it follows from the adjunction that $\mathrm{Hom}(\beta X, [0,1]) \longrightarrow \mathrm{Hom}(X, [0,1])$ is an isomorphism from which one easily sees, from $C^*(X) = \operatorname{colim} \mathrm{Hom}(X, [-n, n])$, that $C^*(\beta X) \longrightarrow C^*(X)$ is an isomorphism. The book [Gillman & Jerison (1960)] is devoted to studying the properties of the C and C^* (contravariant) functors. We recall that a subset of X is called a **zeroset** (respectively, **cozeroset**) if it is the set of all points on which some real-valued function vanishes (respectively, is non-zero).

Before continuing, it will be useful to have some definitions. If a space X is embedded into a space Y, we say that X is **C-embedded** (respectively **C^*-embedded**) in Y

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provided every function (respectively bounded function) on X extends to Y. These definitions are equivalent to saying that $C(Y) \longrightarrow C(X)$ (respectively $C^*(Y) \longrightarrow C^*(X)$) is surjective. An embedding $X \longrightarrow Y$ is called **$\mathcal{CR}$-epic** if the induced map $C(Y) \longrightarrow C(X)$ is an epimorphism in the category of commutative unitary rings. A space is called **absolute $\mathcal{CR}$-epic** if every embedding into a larger space is $\mathcal{CR}$-epic. If $X \longrightarrow K$ is a dense embedding and K is compact, we say that the inclusion $X \hookrightarrow K$ (or, by abuse of notation, K alone) is a **compactification** of X.

We will see in 2.4 below that a necessary and sufficient condition that a space be absolute $\mathcal{CR}$-epic is that every **compactification**, that is every dense embedding into a compact space is $\mathcal{CR}$-epic.

We have explored in several earlier papers the concept of $\mathcal{CR}$-epic embeddings and absolute $\mathcal{CR}$-epic spaces, [Barr, *et al.* (2003), Barr, *et al.* (2005), Barr, *et al.* (2007)]. In this paper we explore several classes of such spaces that generalize examples from the earlier paper. For example, in [Barr, *et al.* (2007)], we looked at the Tychonoff plank. In this paper, we consider the more general notion of deleted planks, see Section 3. These deleted planks give us further examples of absolute $\mathcal{CR}$-epic spaces which are neither Lindelöf nor almost compact. Before outlining the other topics we consider, we will review the definition of CNP spaces, and note that Alster spaces proved very useful in our previous paper as reviewed in Section 2.

A space X is said to satisfy the **Countable Neighbourhood Property** (CNP) if a countable intersection of βX-neighbourhoods of X is a neighbourhood of X. We saw in [Barr, *et al.* (2007), Corollary 3.4] that every Lindelöf CNP space is absolute $\mathcal{CR}$-epic. Since a locally compact space is open in any compactification [Gillman & Jerison (1960), 3.15D], it follows immediately that every locally compact Lindelöf space has the CNP and hence, if Lindelöf, is absolute $\mathcal{CR}$-epic.

Section 3 deals with several generalizations of [Barr, *et al.* (2007), 7.15] in which it was shown that the Dieudonné plank was absolute $\mathcal{CR}$-epic, even though it was not Lindelöf. Section 4 deals with properties of Alster spaces that were derived in [Barr, *et al.* (2007)] under additional, unnecessary, conditions. Section 5 shows that countable unions of Lindelöf absolute $\mathcal{CR}$-epic spaces are absolute $\mathcal{CR}$-epic. Section 6 describes counterexamples to certain plausible conjectures. Sections 7 and 9 give alternate characterizations of the CNP property, while Section 8 looks at spaces of the form $\beta X - X$.

1.1. ERRATA. We would like to mention two errors, one significant one not, in [Barr, *et al.* (2007)]. We claimed in the introduction to that paper that a perfect quotient of an absolute $\mathcal{CR}$-epic space was absolute $\mathcal{CR}$-epic. But we proved only that a perfect quotient of a Lindelöf CNP space is Lindelöf CNP. In Section 6 we give an example that shows that the claim is false.

The second is that we stated a weaker form of the converse part of Theorem 2.8, Smirnov's theorem ([Smirnov (1951)]), than is true or that we actually proved. The correct statement should have been:

If a Tychonoff space X is Lindelöf, then in any compactification K of X any open subset of K that contains X contains a cozeroset containing X. Conversely, if there is a *compactification K of X with the property that every open subset of K that contains X contains a cozeroset containing X, then X is Lindelöf (and then the property holds for every compactification).*

The proof of the converse part is correct if you replace every instance of βX by K.

2. General results

This section consists partly of material from [Barr, *et al.* (2007)]. The reason we have included it is that its contents, both definitions and theorems, are called upon frequently in the present paper. Since length is no longer a consideration in electronic publications, it seems like a good idea to make it as self-contained as feasible. See the original paper for proofs.

2.1. Standard definitions and notation. It is shown in [Gillman & Jerison (1960), Chapter 6] that βX is characterized as the unique compact space in which X is dense and C^*-embedded. We denote by υX the Hewitt realcompactification of a Tychonoff space X; see Chapter 8 of [Gillman & Jerison (1960)] or 5.5(c) and 5.10 of [Porter & Woods (1988)]. A space X is called **realcompact** if $X = \upsilon X$. This space is characterized as the largest subspace of βX to which *every* real-valued function (including the unbounded ones) can be extended.

A continuous map $\theta : X \longrightarrow Y$ is said to be **perfect** if it is closed and for all $p \in Y$, $\theta^{-1}(p)$ is compact. It can be shown that whenever $K \subseteq Y$ is compact, so is $\theta^{-1}(K)$. The properties of perfect maps are explored in detail in [Porter & Woods (1988)]. However, be warned that they do not invariably assume that their functions are continuous, while we do.

In any topological space, a countable intersection of open sets is called a **G_δ-set**. A cover by G_δ sets will be called a G_δ-cover. A subset A of a space X will be called a **P-set** in X provided every G_δ-set that contains A contains an open set containing A. If $A = \{x\}$ is a singleton, then we will say that x is a **P-point**. If every element of X is a P-point, then X is called a **P-space**. We say that a cover of a space by not-necessarily-open sets is **ample** if each compact subset of the space is covered by finitely many of the sets. We say that a space is *amply Lindelöf* if every ample G_δ cover contains a countable subcover. We have subsequently discovered that an equivalent condition was first defined and exploited by K. Alster, [Alster (1988), Condition $(*)$] in connection with the question of which spaces have a Lindelöf product with every Lindelöf space. Accordingly, we will rename this as Alster's condition. (It must be admitted that we were never happy with "amply Lindelöf" in the first place.) A space that satisfies Alster's condition will be called an **Alster space**. We will continue to call a cover of a space *ample* if every compact set is finitely covered. Since G_δ sets are closed under finite unions, we may and often will assume, without loss of generality, that our G_δ covers are closed under finite unions. For an ample G_δ cover, this means that every compact set is contained in some element of the cover.

We have studied Alster spaces in a related paper that concentrated on Alster's original context, namely that the product of an Alster space and a Lindelöf space is Lindelöf. See [Barr, *et al.* (2007a)].

2.2. NOTATION. If $\theta : B \longrightarrow A$ is a function, then we use the same θ for the direct image function $\mathcal{P}(B) \longrightarrow \mathcal{P}(A)$. This has a right adjoint $\theta^{-1} : \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$ and θ^{-1} itself has a right adjoint $\theta_\# : \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$ that takes a set $T \subseteq B$ to $\theta_\#(T) = A - \theta(B - T)$. It follows that if θ is a closed mapping between topological spaces, then $\theta_\#$ takes open sets to open sets, a fact that will turn out to be important. Here are some properties of the adjunctions. Assume that $S \subseteq A$ and $T \subseteq B$. Then

1. $\theta(T) \subseteq S$ if and only if $T \subseteq \theta^{-1}(S)$;
2. $\theta^{-1}(S) \subseteq T$ if and only if $S \subseteq \theta_\#(T)$;
3. θ preserves unions, $\theta_\#$ preserves intersections and θ^{-1} preserves both;
4. $\theta^{-1}(\theta_\#(T)) \subseteq T$, with equality when θ is injective;
5. $\theta_\#(\theta^{-1}(S)) \supseteq S$ with equality when θ is surjective;
6. $\theta(\theta^{-1}(S)) \subseteq S$, with equality when θ is surjective;
7. $\theta^{-1}(\theta(T)) \supseteq T$ with equality when θ is injective;
8. $\theta_\#(T) \subseteq \theta(T)$ if and only if θ is surjective;
9. $\theta^{-1}(\theta(\theta^{-1}(S))) = \theta^{-1}(\theta_\#(\theta^{-1}(S))) = \theta^{-1}(S)$;
10. $\theta(\theta^{-1}(\theta(T))) = \theta(T)$;
11. $\theta_\#(\theta^{-1}(\theta_\#(T))) = \theta_\#(T)$.

Incidentally, $\theta_\#$ is called the universal image in topos theory and usually denoted $\forall_\theta$. In contrast, the direct image is called the existential image and denoted $\exists_\theta$.

We can use this to give an efficient proof of the following well-known folklore.

2.3. PROPOSITION. *A perfect preimage of a Lindelöf space is Lindelöf.*

PROOF. Let $\theta : Y \longrightarrow X$ be a perfect map and suppose that X is Lindelöf. Let $\mathcal{U}$ be an open cover of Y. We can suppose without loss of generality that $\mathcal{U}$ is closed under finite unions. If $p \in X$, then $\theta^{-1}(p)$ is compact, hence there is some $U \in \mathcal{U}$ with $\theta^{-1}(p) \subseteq U$, whence $p \in \theta_\#(U)$. Thus $\{\theta_\#(U) \mid U \in \mathcal{U}\}$ is an open cover of X and has a countable refinement, say $\{\theta_\#(U_n)\}$. Then $\{\theta^{-1}(\theta_\#(U_n))\}$ is an open cover of Y. Since $\theta^{-1}(\theta_\#(A)) \subseteq A$ for any $A \subseteq Y$, we see that $\{\theta^{-1}(\theta_\#(U_n))\}$ refines $\mathcal{U}$. ∎

If $\{X_i \mid i \in I\}$ are disjoint open subspaces of X for which $X = \bigcup_{i\in I} X_i$ we will say that X is the **sum** of the X_i and write $X = \sum_{i\in I} X_i$. We will also write $X = X_1 + \cdots + X_n$ for a finite sum. This is the sum in the category of topological spaces and continuous maps. The sum is called "free union" be topologists.

2.4. THEOREM. *A space is absolute $\mathcal{CR}$-epic if and only if every dense embedding into a compact space is $\mathcal{CR}$-epic.*

PROOF. Suppose $\sigma : X \hookrightarrow Y$ is an embedding. We have a commutative diagram of embeddings

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\tau"'] & Y \arrow[d, "\varphi"] \\ \mathrm{cl}_{\beta Y}(X) \arrow[r, "\psi"'] & \beta Y \end{tikzcd}$$

Apply the contravariant functor C to a commutative diagram

$$\begin{tikzcd} C(X) & C(Y) \arrow[l, "C(\sigma)"'] \\ C(\mathrm{cl}_{\beta Y}(X)) \arrow[u, "C(\tau)"] & C(\beta Y) \arrow[l, "C(\psi)"] \arrow[u, "C(\varphi)"'] \end{tikzcd}$$

The map $C(\varphi)$ is always an epimorphism since $Y \longrightarrow \beta Y$ is a C^*-embedding ([Barr, *et al.* (2003), Proposition 2.1(i)]) and $C(\psi)$ is surjective (hence an epimorphism) since $\mathrm{cl}_{\beta Y}(X)$ is a closed subspace of the normal space βY. Then from elementary category theory, we see that $C(\sigma)$ is epic if and only if $C(\tau)$ is. ∎

2.5. QUOTIENTS. Since every Hausdorff compactification of a space X is a quotient space of βX modulo a closed equivalence relation, we will begin by looking at equivalence relations. Although the results are stated for βX, they are actually valid for any compactification of X.

2.6. DEFINITION. *Let X be a space. An equivalence relation $E \subseteq \beta X \times \beta X$ will be called* **admissible** *if it is a closed subspace of $\beta X \times \beta X$ and if $(X \times \beta X) \cap E = \Delta_X$ (the diagonal of X in $X \times X$).*

Throughout this paper, E will denote an admissible equivalence relation on the Stone-Čech compactification of a space, usually X, and $\theta : \beta X \longrightarrow \beta X/E = K$ will denote the induced quotient map. The map θ, being continuous between compact sets, is closed. It is an immediate consequence that $\theta_\#$ (see 2.2) takes open sets to open sets. Since θ is surjective, $\theta_\#(U) \subseteq \theta(U)$, so that when U is a βX-neighbourhood of X, both $\theta_\#(U)$ and $\theta(U)$ are K-neighbourhoods of $\theta_\#(X)$ and the admissibility of E implies that $\theta_\#(X) = X$.

2.7. Proposition. *For any equivalence relation E on βX, the induced map $\theta : X \longrightarrow K = \beta X/E$ is an embedding into a Tychonoff space if and only if E is admissible.*

For the proof, see [Barr, *et al.* (2007), Proposition 2.5].

Central to our studies is the following theorem, which is essentially [Barr, *et al.* (2005), Corollary 2.14].

2.8. Theorem. *A Lindelöf space X is absolute $\mathcal{CR}$-epic if and only if, for every compactification K of X, every function in $C(X)$ extends to a K-neighbourhood of X.* ■

2.9. Lemma. *Suppose E is an admissible equivalence relation on βX. Then for any $f \in C(\beta X)$ and any $n \in \mathbf{N}$, the set*

$$U_n = \{p \in \beta X \mid (p,q) \in E \Rightarrow |f(p) - f(q)| < 1/n\}$$

is open in βX and contains X.

For the proof, see [Barr, *et al.* (2007), Lemma 2.7].

We will require both halves of the following result from general topology. Although it is well known, we did not find a readily accessible proof and so we sketch one.

2.10. Theorem. [Smirnov] *If a Tychonoff space X is Lindelöf, then in any compactification K of X any open subset of K that contains X contains a cozeroset containing X. Conversely, if every open subset of βX that contains X contains a cozeroset containing X, then X is Lindelöf.*

3. Some (punctured) planks are absolute $\mathcal{CR}$-epic

In [Barr, *et al.* (2007), 7.15], we showed that the Tychonoff plank was absolute $\mathcal{CR}$-epic. In this section, we extend this result to certain "generalized planks".

3.1. Oscillation. We begin with a more detailed look at what is needed in order to extend a real-valued function from a subspace to a point of the containing space. Suppose X is densely embedded in Y, $p \in Y - X$, and $f \in C(X)$. If U is a subset of Y, define

$$\mathcal{O}(f,U) = \sup\{|f(x) - f(x')| \mid x,\, x' \in U \cap X\}$$

called the **oscillation of f on U**. Define

$$\mathcal{O}(f,p) = \inf\{\mathcal{O}(f,U) \mid U \text{ is a } Y\text{-neighbourhood of } p\}$$

called the **oscillation of f at p**.

3.2. Proposition. *Let X be dense in Y, $f \in C(X)$ and $p \in Y - X$. A necessary and sufficient condition that f be extendible continuously to p is that $\mathcal{O}(f,p) = 0$.*

Proof. The necessity of that condition is obvious. So suppose that $\mathcal{O}(f,p) = 0$. For each positive integer n, there is a neighbourhood U_n of p such that x, $x' \in U \cap X$ implies that $|f(x) - f(x')| < 1/n$. Choose an arbitrary element $x_n \in U_n$. It is clear that the sequence $f(x_n)$ is Cauchy and converges to a $t \in \mathbf{R}$. Define $f(p) = t$. The proofs that this construction does not depend on the choice of the x_n and continuously extends f to p are routine and are left to the reader. ∎

3.3. Proposition. *Let X be dense in Y. Let $A = \{y \in Y \mid f \text{ extends to } y\}$. Then the extension is continuous on A.*

Proof. Let us continue to call the extension f. The construction of the preceding proposition implies that if U is a Y-open neighbourhood of $y \in A$, then for any $x \in X \cap U$, $|f(x) - f(y)| \le \mathcal{O}(f, X \cap U)$. Thus if $y' \in U$, we also see that $|f(x) - f(y')| \le \mathcal{O}(f, X \cap U)$. Thus $|f(y) - f(y')| \le 2\mathcal{O}(f, X \cap U)$. It follows that $\mathcal{O}(f, A \cap U) \le 2\mathcal{O}(f, X \cap U)$. Then $\mathcal{O}(f, X \cap U)$ can be made arbitrarily small for appropriate choice of U and hence so can $\mathcal{O}(f, A \cap U)$. This implies that the extended function is continuous on A. ∎

A topological space that is a product of two spaces, often total orders, is sometimes called a **plank**. Sometimes the word is used to denote what we call a **punctured plank**, a product with one point removed. For example, let L denote the one-point Lindelöfization of the first uncountable ordinal, that is $\omega_1 \cup \{\omega_1\}$ with the points of ω_1 isolated and a neighbourhood of ω_1 consisting of any set containing it whose complement is countable. The **Dieudonné plank** is the space $(\omega + 1) \times L - \{(\omega, \omega_1)\}$. In [Barr, *et al.* (2007), 7.14, 7.15] we showed that the Dieudonné plank and certain other punctured planks are absolute $\mathcal{CR}$-epic. Here we extend that result.

We will be dealing with complete lattices that are products of a finite number of complete chains (total orders). Each such lattice has a top and a bottom element that we will denote by $\top$ and $\bot$, respectively. The topology on a chain is given by letting closed intervals be a subbasis of closed sets and the product topology is used for products.

3.4. Proposition. *A complete chain is compact Hausdorff.*

Proof. If X is a complete chain, then for any two elements $x < y$, suppose first that the open interval (x, y) is empty. In this case $[\bot, x]$ and $[y, \top]$ are disjoint clopen sets containing x and y respectively. If there is a $p \in (x, y)$, then $[\bot, p)$ and $(p, \top]$ are disjoint open neighbourhoods of these points. It follows from [Kelley (1955), Theorem 5.6] that to establish compactness, it is sufficient to show that any family of closed intervals with the finite intersection property has non-empty intersection. Suppose we have such a family $\{[x_\alpha, y_\alpha]\}$. Let $x = \sup x_\alpha$ and $y = \inf y_\alpha$. It will suffice to show that $x \le y$. If not, there will be some α for which $x \not\le y_\alpha$ and similarly, some γ with $x_\gamma \not\le y_\alpha$. But then $[x_\gamma, y_\gamma] \cap [x_\alpha, y_\alpha] = \emptyset$, a contradiction. ∎

A space X is **almost compact** if $\beta X - X$ has at most one point. This is equivalent to the fact that of any pair of disjoint zerosets, at least one is compact. A space is **pseudocompact** if every real-valued is bounded. An almost compact space is pseudocompact (see [Gillman & Jerison (1960), Exercise 6J]).

3.5. Proposition. *Suppose X and Y are compact spaces and $x_0 \in X$ and $y_0 \in Y$ are such that $X - \{x_0\}$ and $Y - \{y_0\}$ are almost compact. Then $X \times Y - \{(x_0, y_0)\}$ is almost compact.*

Proof. According to [Gillman & Jerison (1960), 9.14] the product of a compact space and a pseudocompact space is pseudocompact. In particular, $X \times (Y - \{y_0\})$ is pseudocompact. According to [Glicksberg, 1959, Theorem 1], this implies that

$$\beta(X \times (Y - \{y_0\})) = X \times Y$$

Since the β-compactification of any space between Z and βZ is βZ, we conclude in this case that

$$\beta(X \times Y - \{(x_0, y_0))\} = X \times Y \qquad \blacksquare$$

3.6. Proposition. *Suppose that x and y are P-points in X and Y, respectively. Then (x, y) is a P-point of $X \times Y$.*

Proof. If $\{W_n\}$ is a countable family of neighbourhoods of (x, y), there are for each n, neighbourhoods U_n and V_n of x and y, respectively, such that $U_n \times V_n \subseteq W_n$ and then $\bigcap U_n \times \bigcap V_n \subseteq \bigcap W_n$, which is thereby a neighbourhood of (x, y). $\blacksquare$

We will say that a complete chain X **has (respectively lacks) a proper countable cofinal subset** when $X - \{\top\}$ does. Clearly X lacks a countable cofinal subset if and only if $X - \{\top\}$ is countably complete.

3.7. Theorem. *Let X be a finite product of complete chains, each which lacks a proper countable cofinal subset. Then $X - \{\top\}$ is almost compact and $\top$ is a P-point of X.*

Proof. First suppose X is a chain that lacks a proper countable cofinal set. We claim that $\beta(X - \{\top\}) = X$. Since the β-compactification of a space is characterized as the unique compactification in which X is C^*-embedded, this amounts to showing that every bounded function on $X - \{\top\}$ extends continuously to $\top$. Suppose $f : X - \{\top\} \longrightarrow [0, 1]$. We must show that $\mathcal{O}(f, \top) = 0$. If not, there is some $\epsilon > 0$ such that for every neighbourhood U of $\top$, there are elements x, $x' \in U - \{\top\}$ for which $|f(x) - f(x')| > \epsilon$. A neighbourhood base at $\top$ consists of sets of the form $(x, \top]$. Choose one such neighbourhood, say $(x_0, \top]$ and choose x_1, $x_1' \in (x_0, \top)$ so that $f(x_1) - f(x_1') > \epsilon$. Now choose elements $x_2, x_2' \in (x_1 \vee x_1', \top)$ so that $f(x_2') - f(x_2) > \epsilon$ and so on by induction. We get two increasing sequences $x_1 < x_2 < \cdots < x_n < \cdots$ and $x_1' < x_2' < \cdots < x_n' < \cdots$ such that $x_n < x_{n+1}'$, $x_n' < x_{n+1}$ and $f(x_n) - f(x_n') > \epsilon$. We see, in particular, that $x_{n-1} < x_n' < x_{n+1}$ for all n, which implies that $x = \sup x_n = \sup x_n'$. The fact that X lacks a countable cofinal subset implies that $x \neq \top$. Next we claim that the open intervals containing a point constitute a neighbourhood at that point. In fact, the complement of an open interval consists of at most two closed intervals and is therefore open. On the other hand, the complement of a finite union of closed intervals is the union of a finite set of open intervals, at most one of which will contain any given point. If there were a neighbourhood (x, x') that contained none of the x_n', then x' would be a smaller upper bound for the sequence of x_n

and similarly for the x'_n. Thus the fact that $f(x_n) - f(x'_n) > \epsilon$ contradicts the fact that f is well-defined and continuous at x.

If $\{U_n\}$ is a countable family of neighbourhoods of $\top$, we can assume they have the form $(x_n, \top]$, whose intersection is $[\sup x_n, \top]$. Since $\sup x_n < \top$, we see that $(\sup x_n, \top]$ is a open neighbourhood of $\top$.

An obvious induction based on the previous two propositions extends the conclusion from chains to finite products of them. ∎

Now we turn to the case of spaces that do have countable cofinal sets.

3.8. Theorem. *Suppose that $X = \prod X_i$ is a product of finitely many complete chains, each of which has a countable cofinal set. Then $X - \{\top\}$ is locally compact and σ-compact.*

Proof. The local compactness is obvious. It is obvious that a complete chain with a countable cofinal subset is σ-compact, so it suffices to show that if X and Y are spaces and $x \in X$ and $y \in Y$ are such that $X - \{x\}$ and $Y - \{y\}$ are σ-compact then the same is true of $X \times Y - \{(x, y)\}$. Since the product and finite sum of σ-compact spaces is σ-compact, the conclusion follows from

$$X \times Y - \{(x,y)\} = ((X - \{x\}) \times (Y - \{y\})) \cup (\{x\} \times (Y - \{y\})) \cup ((X - \{x\}) \times \{y\}) \quad \blacksquare$$

A space is said to be **weakly Lindelöf** if from every open cover a countable subset can be found whose union is dense. Obviously a Lindelöf space is weakly Lindelöf and so is any space with a dense Lindelöf subspace.

3.9. Theorem. *Suppose that Y is a space and y_0 is a non-isolated P-point. Suppose Z is a space with a point z_0 such that $Z - \{z_0\}$ is weakly Lindelöf. Then $Y \times Z - \{(y_0, z_0)\}$ is C-embedded in $Y \times Z$.*

Proof. The case that z_0 is isolated is immediate from 3.7. Otherwise, let $f \in C(Y \times Z - \{(y_0, z_0)\})$. For each $z \in Z - \{z_0\}$, f is continuous at (y_0, z). Thus for each $n \in \mathbf{N}$, there is a neighbourhood $V(z, n)$ of y_0 in Y and a neighbourhood $W(z, n)$ of z in $Z - \{z_0\}$ such that $\mathcal{O}(f, V(z, n) \times W(z, n)) < 1/n$. Suppose $z(1, n), z(2, n), \ldots$ is a countable set of points of $Z - \{z_0\}$ such that $W(n) = \bigcup_{m \in \mathbf{N}} W(z(m, n), n)$ is dense in $Z - \{z_0\}$. Let $V(n) = \bigcap_{m \in \mathbf{N}} V(z(m, n), n)$. It follows that $\mathcal{O}(f, V(n) \times W(n)) \le 1/n$. Since y_0 is a P-point, $V(n)$ is a neighbourhood of y_0. For any $y \in Y - \{y_0\}$ both functions $f(y, -)$ and $f(y_0, -)$ are continuous on $Z - \{z_0\}$ and hence, for any $z \in Z - \{z_0\}$ and any $m \in \mathbf{N}$, there is a neighbourhood $T(m)$ of z such that the oscillation in T_m of both $f(y, -)$ and $f(y_0, -)$ is at most $1/m$. There is some $p \in W(n) \cap T(m)$ and we have

$$\begin{aligned} |f(y,z) - f(y_0,z)| &\le |f(y,z) - f(y,p)| + |f(y,p) - f(y_0,p)| + |f(y_0,p) - f(y_0,z)| \\ &< 1/m + 1/n + 1/m = 2/m + 1/n \end{aligned}$$

Since the left hand side does not depend on m this implies that $|f(y, z) - f(y_0, z)| \le 1/n$. Finally, let $V = \bigcap_{n \in \mathbf{N}} V(n)$. Then for $y \in V$, we have $f(y, z) = f(y, z_0)$ and we can extend f by $f(y_0, z_0) = f(y, z_0)$ for any $y \in V$. ∎

This works, in particular, if $Z - \{z_0\}$ is Lindelöf or if it contains a dense Lindelöf subspace.

From this, we get the following generalization of [Barr, *et al.* (2007), Theorem 7.14].

3.10. Theorem. *Suppose the Lindelöf space Y is the union of a locally compact subspace and a non-isolated P-point y_0. Suppose Z is a compact space that has a proper dense Lindelöf subspace and z_0 is a point not in that subspace. Then $Y \times Z - \{(y_0, z_0)\}$ is absolute $\mathcal{CR}$-epic.*

Proof. Since $D = Y \times Z - \{(y_0, z_0)\}$ is C-embedded in $Y \times Z$, it follows that the realcompactification $\upsilon(D) = Y \times Z$. Since $Y \times Z$ is the union of a locally compact space and a compact space, the result follows from [Barr, *et al.* (2007), Theorem 7.11]. ∎

3.11. Theorem. *Suppose that $X = \prod_{i=1}^{n} X_i$ is a finite product of complete chains. Assume that $\top$ is not an isolated point of any of the chains. Then $X - \{\top\}$ is absolute $\mathcal{CR}$-epic.*

Proof. Divide the spaces into two classes, $Y_1, Y_2, \ldots, Y_k$ that lack proper countable cofinal sets and $Z_1, Z_2, \ldots, Z_l$ that have them. Let $Y = \prod Y_i$ and $Z = \prod Z_j$. We know that $\top$ is a non-isolated P-point of Y. We know that $Z - \{\top\}$ is σ-compact, hence Lindelöf. It follows that $Y \times Z - \{(\top, \top)\}$ is absolute $\mathcal{CR}$-epic. ∎

4. Alster's condition

Recall from the introduction that a space is an Alster space if every ample cover by G_δ-sets has a countable refinement.

Most of the following theorem was proved as [Barr, *et al.* (2007), Theorem 4.7] under the additional hypothesis that the spaces satisfied the CNP. That condition was not used in the proofs; it was simply that we never studied Alster's condition separately.

4.1. Theorem.

1. *The product of two Alster spaces is Alster space.*
2. *A closed subspace of an Alster space is Alster space.*
3. *If $t : Y \longrightarrow X$ is a continuous surjection and Y satisfies Alster's condition, so does X.*
4. *A union of countably many Alster spaces is an Alster space.*
5. *A Lindelöf space is an Alster space if every point has a neighbourhood that satisfies Alster's condition.*
6. *A cozero-subspace of an Alster space is an Alster space.*
7. *If $\theta : Y \longrightarrow X$ is a perfect surjection (see 2.1) and X satisfies Alster's condition so does Y.*

Proof. We will mention briefly what, if any changes are needed from the proofs in [Barr, *et al.* (2007), Theorem 4.7].

1. This is just [Barr, *et al.* (2007), Theorem 4.5].

2. See [Barr, *et al.* (2007), proof of 4.7.2]

3. Let $\mathcal{U}$ be an ample G_δ cover of X. Since θ^{-1} preserves open sets and commutes with meets, it follows that $\theta^{-1}(\mathcal{U}) = \{\theta^{-1}(U) \mid U \in \mathcal{U}\}$ is a G_δ cover of Y. It is also ample since if K is a compact subset of Y, $\theta(K)$ is a compact subset of X and is therefore finitely covered by $\mathcal{U}$. But then $\theta^{-1}(\theta(K))$, which contains K, is finitely covered by $\theta^{-1}(\mathcal{U})$. It follows that Y is covered by some countable subset $\{\theta^{-1}(U_n)\}$ and it is clear that $\{U_n\}$ covers X.

4. A countable sum Alster spaces is obviously an Alster space and a union is a continuous image of a sum, so the preceding item finishes it.

5. Immediate from the preceding item.

6. Immediate from the second item and the preceding one since cozerosets are F_σ.

7. The proof of [Barr, *et al.* (2007), 4.7.5] does not use the CNP condition. However the first part of the proof contains an unneeded gap, so we give a complete proof here. Suppose that $\theta : Y \longrightarrow X$ is perfect and X satisfies Alster's condition. Let $\mathcal{V}$ be an ample cover of Y. We may assume that $\mathcal{V}$ is closed under finite unions. If $K \subseteq X$ is compact—possibly a singleton—then $\theta^{-1}(K)$ is compact and hence there is a $V \in \mathcal{V}$ with v$\theta^{-1}(K) \subseteq V$ and hence $K = \theta_{\#}(\theta^{-1}(K)) \subseteq \theta_{\#}(V)$. This implies that $\mathcal{U} = \{\theta_{\#}(V) \mid V \in \mathcal{V}\}$ is an ample cover of X and hence contains a finite subcover, say $\mathcal{U}_0$. Clearly $\mathcal{V}_0 = \{\theta^{-1}(U) \mid U \in \mathcal{U}_0\}$ is a countable cover of Y. Since $\theta^{-1}(t_{\#}(V)) \subseteq V$, we see that $\mathcal{V}_0$ refines $\mathcal{V}$. ■

4.2. Theorem. *Suppose that the Lindelöf space X has the property that, for each $p \in X$, whenever V is a G_δ set of βX that contains p, then $V \cup X$ is a βX-neighbourhood of p. Then X is a CNP space that satisfies Alster's condition.*

Proof. CNP is immediate. Let $\mathcal{U}$ be an ample cover of X by G_δ sets of βX. We can suppose, without loss of generality, that $\mathcal{U}$ is closed under finite union. Let p be a point of X. We will show that there is a βX-neighbourhood of p that is covered by a countable subset of $\mathcal{U}$. Let $p \in U \in \mathcal{U}$. We claim there is a continuous $f : \beta X \longrightarrow [0, 1]$ such that $p \in \mathcal{Z}[f] \subseteq U$. In fact, this is immediate when U is open and it follows for G_δ sets since a countable intersection of zerosets is a zeroset. Since $V = \mathcal{Z}[f]$ is a G_δ of βX containing p, the hypothesis implies that $X \cup V$ is a βX-neighbourhood of p. Thus there is a compact βX-neighbourhood W of p with $W \subseteq X \cup V$. It follows that $W \cap \operatorname{coz}(f) \subseteq X$. But $W \cap \operatorname{coz}(f)$ is σ-compact and hence there are $U_1, U_2, \ldots \in \mathcal{U}$ such that $W \cap \operatorname{coz}(f) \subseteq \bigcup U_n$ and then $W \subseteq U \cup \bigcup U_n$. Having done this for each point $p \in X$, the Lindelöf property implies that there are countably many points for which the corresponding sets of W cover X and then so do the corresponding sequences of U_n. ■

For example, this theorem implies that when $p \in \beta\mathbf{N} - \mathbf{N}$ is a P-point, then $\mathbf{N} \cup \{p\}$ is absolute $\mathcal{CR}$-epic. See also [Barr, *et al.* (2007), Theorem 5.4].

5. Countable unions of Lindelöf absolute $\mathcal{CR}$-epic spaces

5.1. Proposition. *A countable union of Lindelöf absolute $\mathcal{CR}$-epic subspaces whose interiors cover it, is absolute $\mathcal{CR}$-epic.*

Proof. Let $X = \bigcup X_n = \bigcup \mathrm{int}_X(X_n)$. Since X is a countable union of Lindelöf spaces, it is Lindelöf. Let K be a compactification of X and $f \in C^*(X)$. Let $U_n = \mathrm{int}_X(X_n)$ and $K_n = \mathrm{cl}_K(X_n)$. Since K_n is a compactification of X_n, $f|X_n$ extends to a K_n-open subset $W_n \subseteq K_n$ that contains X_n. Since U_n is open in X there is a K-open set V_n such that $X \cap V_n = U_n$. It is well known, as X is dense in K, that $V_n \subseteq \mathrm{cl}_K(U_n) \subseteq \mathrm{cl}_K(X_n) = K_n$. We claim that $W_n \cap V_n$ is open in K. In fact, $W_n \cap V_n$ is open in K_n, hence open in the subset V_n, while V_n is open in K. Since $f|X_n$ extends to W_n, the conclusion follows from 3.3.

5.2. Theorem. *The classes of Lindelöf absolute $\mathcal{CR}$-epic spaces, Lindelöf CNP spaces, and Alster spaces are all closed under countable open unions.*

Proof. The first case is included in the preceding proposition. The second is essentially [Barr, *et al.* (2007), Theorem 4.7.3] once we note that a countable union of Lindelöf spaces is Lindelöf. The case of Alster spaces follows from Theorem 4.1.4. ∎

It is trivial to extend these conclusions to covers whose interiors also cover. We omit the details.

5.3. Theorem. *A countable, locally finite union of closed Lindelöf CNP (respectively Alster) spaces is Lindelöf CNP (respectively Alster).*

Proof. By [Barr, *et al.* (2007), Theorem 4.7.4], a countable sum of Lindelöf CNP spaces is Lindelöf CNP and we show in [Barr, *et al.* (2007a)] that CNP spaces are closed under perfect image (and it is well known that being a Lindelöf space is as well.) Thus it is sufficient to show that the map from the sum to the union is perfect. The inverse image of each point is finite, hence compact. Let $A = \sum A_n$ be a closed subset of the sum with A_n closed in X_n. If $p \notin \bigcup A_n$ there is a neighbourhood U of p that meets only finitely many of the X_n, say $X_1, X_2, \ldots, X_m$. For each $n \le m$, the set A_n is closed in X_n, which is closed in X and hence $\bigcup_{n=1}^m A_n$ is closed in X. Since $p \notin \bigcup_{n=1}^m A_n$, there is a neighbourhood V of p which misses that union. Since U does not meet any X_n for $n > m$, neither does $V \cap U$ so there is a neighbourhood of p that does not meet A. Since p was an arbitrary point not in A, we conclude that A is closed. ∎

By contrast, we will see in the next section that even a finite union of closed Lindelöf absolute $\mathcal{CR}$-epic, but not CNP spaces, need not be absolute $\mathcal{CR}$-epic.

6. Three counter-examples

In this section we show by example that a Lindelöf absolute $\mathcal{CR}$-epic space need not satisfy the CNP, that a perfect quotient of a Lindelöf absolute $\mathcal{CR}$-epic space need not be absolute $\mathcal{CR}$-epic, and that a Lindelöf space need not be absolute $\mathcal{CR}$-epic even if it is the union of two closed absolute $\mathcal{CR}$-epic subspaces.

6.1. The space X. Let $\{X_n\}$ be any countable family of non-compact absolute $\mathcal{CR}$-epic Lindelöf spaces. We can picture the space $\beta\left(\sum X_n\right)$ as the union of three disjoint subspaces:

<table>
<tr><td colspan="2">$A = \beta(\sum X_n) - \sum \beta X_n$</td></tr>
<tr><td>$B = \sum X_n$</td><td>$C = \sum(\beta X_n - X_n)$</td></tr>
</table>

We let X be the space $A \cup B$. Since X lies between $\sum X_n$ and its β-compactification, we see that $\beta X = \beta\left(\sum X_n\right)$ and therefore $\beta X - X = C$.

Since X_n is a summand of $\sum X_n$, βX_n is a summand of $\beta\left(\sum X_n\right)$, so that $\sum \beta X_n$ is open in $\beta\left(\sum X_n\right)$ which implies that A is compact. But then X is the union of a Lindelöf space and a compact space and is therefore Lindelöf.

6.2. X is absolute $\mathcal{CR}$-epic, but does not satisfy the CNP. We define the **level function** $\ell : \beta X \longrightarrow \mathbf{N} \cup \{\infty\}$, the one point compactification of $\mathbf{N}$, as the continuous extension of the function that has the value n on X_n. It is clear that $\ell(p) = n$ if and only if $p \in \beta X_n$ and $\ell(p) = \infty$ otherwise.

Let E be an admissible equivalence relation on βX. This means that $\beta X/E$ is Hausdorff and that X is embedded in it, see 2.6. Let $\theta : \beta X \longrightarrow \beta X/E$ be the canonical projection. We will say that an element p is **E-equivalent** to an element q if $(p, q) \in E$, which is the same as $\theta(p) = \theta(q)$.

6.3. Lemma. *There is an $N \in \mathbf{N}$ such that whenever p is E-equivalent to q, then either $p = q$ or $\ell(p)$, $\ell(q) < N$.*

Proof. Suppose we can find elements of arbitrarily high level that are E-equivalent to elements other than themselves. We will consider two cases. First suppose we can find a sequence of elements $(p_n, q_n) \in E$ such that the levels of the p_n are unbounded while those of the q_n are bounded. Restricting to a subsequence, if necessary, we can assume that for some $N \in \mathbf{N}$ we have $\ell(q_n) \leq N$ for all n, while $N < \ell(p_1) < \ell(p_2) < \cdots$. The sequence $\{(p_n, q_n)\} \subseteq E$ thus constructed is discrete since the $\beta X_{\ell(p_n)} \times \beta X_{\ell(q_n)}$ are a family of disjoint open sets of $\beta X \times \beta X$ each containing one element of the set. But the sequence has a limit point (p, q) and it is clear that $\ell(p) = \infty$, which implies that $p \in X$, while $\ell(q) \le N$. The result is that an element of X is E-equivalent to some other element of βX (which might or might not belong to X) which contradicts the fact that E is admissible.

In the other case, there are pairs $(p_n, q_n) \in E$ in which $p_n \ne q_n$ and both $\ell(p_n)$ and $\ell(q_n)$ are unbounded. In that case, proceed as above, but assume that we have chosen subsequences so that $\ell(p_{n-1}) \vee \ell(q_{n-1}) < \ell(p_n) \wedge \ell(q_n)$. Again the sequence $\{(p_n, q_n)\}$ is discrete and hence has a limit point (p, q) but now both elements belong to X and we must show that $p \ne q$. We do this as follows. Since $p_1 \ne q_1$, there is a function f_1, defined on all the elements $\sum \beta X_n$ of levels up to $\ell(p_1) \vee \ell(q_1)$ such that $f_1(p_1) = 0$ and $f_1(q_1) = 1$. Since the levels of both p_2 and q_2 are above those of both p_1 and q_1, this function can be extended to a function f_2 defined on the summands of levels up to $\ell(p_2) \vee \ell(q_2)$ in such a way that $f_2(p_2) = 0$ and $f_2(q_2) = 1$. Continuing in this way, we define a function f on the sum of the finite levels such that $f(p_n) = 0$ and $f(q_n) = 1$. This function has a unique extension to the elements at infinite level and it is clear that $f(p) = 0$, while $f(q) = 1$. Again, this contradicts the admissibility of E. ∎

6.4. Theorem. *The space X is absolute $\mathcal{CR}$-epic, but does not satisfy the CNP.*

Proof. We will use [Barr, *et al.* (2007a), Theorem 2.4]. Let E be an admissible equivalence relation on βX so that $K = \beta X/E$ is a compactification of X with canonical projection $\theta : \beta X \longrightarrow K$. For a fixed integer N, let $L_N = \{p \in \beta X \mid \ell(p) \le N\}$ and $U_N = \{p \in \beta X \mid \ell(p) > N\}$. Then βX is the topological sum $L_N + U_N$. Let N be as in the preceding lemma. This forces θ to respect the decomposition so that $K = \theta(L_N) + \theta(U_n)$. Let $f : X \longrightarrow [0, 1]$ be a given continuous function. We must show that f extends to an open subset of K by showing that it extends to an open subset of $\theta(L_N)$ and to an open subset of $\theta(U_N)$. Since $\sum_{n=1}^{N} X_n$ is absolute $\mathcal{CR}$-epic the restriction of f to that set extends to an open set in $\sum_{n=1}^{N} \beta X_n$, which is open in $\sum_{n=1}^{\infty} \beta X_n$. Since there are no pairs of distinct E-equivalent elements in $\sum_{n>N} \beta X_n$, the restriction of f to elements of X of level above N extends to that open set. Thus f extends to an open set in $K = \beta X/E$ that contains X. Since X is Lindelöf, it follows that X is absolute $\mathcal{CR}$-epic.

To see that X is not CNP, choose a sequence of elements $S = \{p_n\}$ such that $p_n \in \beta X_n - X_n \subseteq \beta X - X$. Let $U_n = \beta X - \{p_n\}$. Then each U_n is a neighbourhood of X in βX. But $\bigcap U_n$ is not a neighbourhood of X since no point in the set $D = \mathrm{cl}_{\beta X}(S) - S$ has a neighbourhood in the intersection. Note that since ℓ is continuous, each point in D is at level ∞ and is therefore an element of X. ∎

6.5. A perfect quotient of X need not be absolute $\mathcal{CR}$-epic. Let S and D be as in the preceding paragraph. Each p_n has a neighbourhood, namely βX_n, that contains no other element of S so that D is closed and, as shown above, $D \subseteq X$. We then show that:

6.6. Proposition. *If D is as above, the quotient space $Y = X/D$ gotten by identifying the points of D to a single point is Tychonoff and the quotient map $\theta : X \longrightarrow Y$ is perfect.*

Proof. The set D is compact in βX, hence also closed in X. Let $W \subseteq X$ be closed (possibly a single point) and $p \in X$ be a point such that $\theta(p) \notin \theta(W)$. If $W \cap D = \emptyset$, there is a function $f : X \longrightarrow [0,1]$ that vanishes on $D \cup \{p\}$ and is 1 on W. If $W \cap D \neq \emptyset$, there is a function f that vanishes at p and is 1 on $W \cup D$. In either case, f is constant on D and so descends to Y. Since every point not in $\theta(W)$ can be separated from it by a function, $\theta(W)$ is closed. Applied to a single point, we see that Y is Hausdorff. ∎

6.7. Corollary. *The space Y is not absolute $\mathcal{CR}$-epic.*

Proof. Let K be the space βX with the points of D identified to a point. There is an obvious dense embedding of Y in K. The sequence $\{p_n\}$, considered as sequence in $K - Y$, converges to the point $\theta(D)$. The conclusion follows from [Barr, *et al.* (2005), Theorem 2.22]. ∎

This gives a second argument that X is not CNP, since a perfect quotient of a CNP space is CNP, [Barr, *et al.* (2007), Theorem 3.6.5].

6.8. A Lindelöf space that is the union of two absolute $\mathcal{CR}$-epic closed subspaces need not be absolute $\mathcal{CR}$-epic. To get this example we use the space X of 6.2, *with the additional assumption that all the X_n are locally compact.* That implies that $\sum X_n$ is locally compact and hence open in $\beta\left(\sum X_n\right)$. We will use $\dot{+}$ to denote a disjoint union of subspaces that is not necessarily a topological sum. As above, we let $X = A \dot{+} B$, whence $\beta X = A \dot{+} B \dot{+} C$. Since A is compact and since each X_n is locally compact, it readily follows that B is open in βX. Since X_n is a topological summand of $\sum X_n$, βX_n is a topological summand of $\beta\left(\sum X_n\right)$ and it follows that $B \dot{+} C = \sum \beta X_n$ is open in $\beta\left(\sum X_n\right)$. Now let X' denote a space disjoint from and homeomorphic to X and let A', B', and C' be the subspaces of $\beta X'$ corresponding to A, B, and C, respectively. We begin with:

6.9. Lemma. *Suppose A can be embedded as a closed subset of the (Tychonoff) spaces X and Y. Then the pushout space $Z = X +_A Y$ is Tychonoff provided either $X = Y$ or one of X and Y is normal.*

Proof. We begin with a closed subset B of Z and a point $p \notin B$. We divide the proof into three cases. The first case, that $p \notin A$, uses neither subsidiary hypothesis. We can suppose, without loss of generality that $p \in X$. Then $p \notin A \cup (B \cap X)$ so that there is a function $f \in C(X)$ such that $f(p) = 0$ and $f = 1$ on $A \cup (B \cap X)$. We can extend f to all of Z by letting $f = 1$ on all Y. Since $f = 1$ on A this is compatible with the previous values. Then $f|X$ is continuous and $f|Y$ is even constant. If $\theta : X + Y \longrightarrow Z$ is the quotient mapping, $\theta^{-1}(X) = X \cup A$ which is closed in $X + Y$ and hence X is closed in Z. Similarly, Y is closed in Z and hence f is continuous on Z.

If $p \in A$ and $X = Y$, then let φ_0 and φ_1 be the two maps of $X \longrightarrow Z$, using the first summand and the second. Then one readily sees that $p \notin \varphi_0^{-1}(B) \cup \varphi_1^{-1}(B)$ and hence there is an $f \in C(X)$ such that $g(p) = 0$ and $g = 1$ on $\varphi_0^{-1}(B) \cup \varphi_1^{-1}(B)$. Let $f \in C(Z)$ be the function such that $f|X = g$ and $f|Y = g$. Since g is the same on both copies of X, this function is well defined and clearly is 1 on B.

Finally, suppose that $p \in A$ and Y is normal. Let $f \in C(X)$ such that $f(p) = 0$ and $f = 1$ on $B \cap X$. Extend this to a continuous function on $X \cup B$ by letting $f = 1$ on all of B. Since this is 1 on $B \cap X$ the extension is continuous as above. Since Y is normal, the function $f|(A \cup (B \cap Y))$ can be extended continuously to all Y (see [Gillman & Jerison (1960), Exercise 3D]) and, as above, the total function is also continuous and is 1 on B. ■

Define $Y = X +_{A=A'} X'$, that is the sum $X + X'$ with each point of A identified with the corresponding point of A'. The square

$$\begin{array}{ccc} A \cong A' & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ X' & \longrightarrow & Y \end{array}$$

is a pushout in the category of topological spaces. From the preceding lemma, we see that the facts that A is closed and X is Lindelöf, hence normal, imply that Y is Tychonoff. Now we can show that Y is not absolute $\mathcal{CR}$-epic. Define $f : Y \longrightarrow [0,1]$ by $f(x) = 1/n$ if $x \in X_n$ while f vanishes on all of X'. This is clearly continuous on the closed sets X and X' and agrees on $X \cap X' = A$ and so is continuous on Y. But it is clear that if f were to extend to a single $y \in C = \sum(\beta X_n - X_n)$, then for $\beta X_n - X_n$ we would have to have $f(y) = 1/n$ on the one hand and $f(y) = 0$ on the other. Thus the maximal extension of f on K is to Y, which is not open in βY because any sequence of points of increasing level in C has all its limit points in A.

7. Characterizations of Lindelöf CNP spaces

In the following theorem, L denotes the convergent sequence $1, 1/2, 1/3, \ldots, 0$.

7.1. THEOREM. *A Lindelöf space satisfies the CNP if and only if its product with L is absolute $\mathcal{CR}$-epic.*

PROOF. The product of a Lindelöf CNP space with a compact space is Lindelöf CNP ([Barr, *et al.* (2007), Theorem 4.6]) and therefore absolute $\mathcal{CR}$-epic, so that direction is trivial. Conversely, assume that $X \times L$ is absolute $\mathcal{CR}$-epic. Suppose $\{U_n\}$ is a countable family of βX-open neighbourhoods of X. We may assume, without loss of generality, that $U_n \supseteq U_{n+1}$. We must show that $U = \bigcap U_n$ is a βX-neighbourhood of X. Define an equivalence relation E on $\beta X \times L$ as follows. Let $A_n = \beta X - U_n$ and E_n be the equivalence relation generated by $((p, 1/n), (p, 0)) \in E_n$ whenever $p \in A_n$. Then let E be the equivalence relation generated by $\bigcup E_n$.

7.2. LEMMA. *The set E is closed in $(\beta X \times L) \times (\beta X \times L)$.*

Proof. The map $\beta X \times L \times L \longrightarrow (\beta X \times L) \times (\beta X \times L)$ that sends (p, s, t) to $((p, s), (p, t))$ is clearly a closed embedding. Let D denote the subset consisting of the elements of the form (p, t, t). It is then readily verified that E is the direct image under this map of the set B that is the union of the following four sets (in which we denote the usual inf in $\mathbf{Z}$ by $\wedge$):

1. $\{(p, 1/n, 1/m) \mid p \in A_{n\wedge m}\}$;
2. $\{(p, 1/n, 0) \mid p \in A_n\}$;
3. $\{(p, 0, 1/n) \mid p \in A_n\}$;
4. D

It suffices to show that B is a closed subset of $\beta X \times L \times L$. It is easily verified that $t = (q, 1/n, 1/m) \in B$ if and only if $n = m$ or $q \in A_{n\wedge m}$. It follows that if $t = (q, 1/n, 1/m) \notin B$, then $U_{n\wedge m} \times \{1/n\} \times \{1/m\}$ is a neighbourhood of t that misses B. Similarly, $s = (q, 1/n, 0) \in B$ if and only if $q \in A_n$, so that if $s \notin B$ then $U_n \times \{1/n\} \times H_n$ is a neighbourhood of s which misses B where $H_n = \{0\} \cup \{1/k \mid k > n\}$. A similar proof works for $(q, 0, 1/n)$ and, since D is obviously closed, it readily follows that B is closed. ■

Proof of 7.1, concluded. Assume that $X \times L$ is absolute $\mathcal{CR}$-epic. We see that $(\beta X \times L)/E$ is a compactification of $X \times L$, see 2.7. Let $f : X \times L \longrightarrow \mathbf{R}$ be the second projection (recall that $L \subseteq [0, 1]$). Since $X \times L$ is absolute $\mathcal{CR}$-epic, f extends to an open set W where $W \subseteq (\beta X \times L)/E$. Consider the map $\beta X \longrightarrow \beta X \times L$ which sends p to $(p, 0)$. Let V be the inverse image of W under this map. Clearly V is an open subset of βX which contains X. It suffices to show that V is contained in $\bigcap U_n$. But if $(p, 0) \in V$ then p must be in $\bigcap U_n$ otherwise $p \in A_n$ for some n and so $((p, 0), (p, 1/n)) \in E$ which shows that f cannot extend to $(p, 0)$ as $f(p, 0) = 0$ but $f(p, 1/n) = 1/n$. ■

7.3. Remark. *Since a closed subspace of a Lindelöf absolute $\mathcal{CR}$-epic space is absolute $\mathcal{CR}$-epic (use [Barr, et al. (2007), Theorem 6.1] in conjunction with the fact that a Lindelöf space is normal and hence any closed subspace is C^*-embedded in it), one readily sees that if L is any space that contains a proper convergent sequence and $X \times L$ is Lindelöf absolute $\mathcal{CR}$-epic, then X is Lindelöf CNP.*

A space X is said to have the **sequential bounded property** or SBP at the point p if for any sequence $\{f_n\}$ of functions in $C(X)$ there is a neighbourhood of p on which each of the functions is bounded. A space has the SBP if it does so at every point. For example, every locally compact space has this property. So does every P-space. The easiest way to see this is to let $p \in X$ and let $U_n = \{x \in X \mid |f_n(p) - f_n(x)| < 1\}$. Then $\bigcap U_n$ is a G_δ containing p and in a P-space, every G_δ is open. Since both of these classes of spaces have the CNP, the following characterization comes as no surprise.

7.4. Theorem. *A Lindelöf space is CNP if and only if it has the SBP at every point.*

PROOF. Suppose X is Lindelöf with the CNP. Let f_1, f_2, ... be a sequence of functions in $C(X)$. We can replace each f_n by $1 + |f_n|$ and assume that they are all positive and bounded away from 0. Then the functions $g_n = 1/f_n$ are all bounded and hence extend to βX. Let U_n be the cozeroset of the extension of g_n to βX. The CNP implies that $\bigcap U_n$ is a neighbourhood of X. Now let $p \in X$. There is a closed, hence compact, βX-neighbourhood V of p inside $\bigcap U_n$. Since every g_n is non-zero on V, it follows that every f_n is bounded there. In particular, every f_n is bounded in $V \cap X$, which is an X-neighbourhood of p.

Conversely, suppose X satisfies the SBP. Let $\{U_n\}$ be a sequence of K-neighbourhoods of X. By Smirnov's theorem 2.10, the Lindelöf property allows us to choose, for each n, a function $g_n : K \longrightarrow [0,1]$ such that $X \subseteq \mathrm{coz}(g_n) \subseteq U_n$. Define $f_n : K \longrightarrow [1,\infty]$ by $f_n(p) = 1/g_n(p)$ when $g_n(p) \neq 0$ and $f_n(p) = \infty$ otherwise. Then $X \subseteq \mathrm{fin}(f_n) \subseteq U_n$, where $\mathrm{fin}(f_n) = \mathrm{coz}(g_n)$ is the set on which f_n is finite. For each $p \in X$, there is an X-open set V_p on which each f_n is bounded, from which it is clear that each f_n is bounded on $W_p = \mathrm{cl}_K(V_p)$, which is a K-neighbourhood of p. Since f_n is bounded on W_p, it follows that $W_p \subseteq \mathrm{fin}(f_n) \subseteq U_n$ for each p and each n, so that $W = \bigcup_{p\in X} V_p$ is a K-neighbourhood of X contained in $\bigcap U_n$. ∎

7.5. DEFINITION. *If f, $g \in C(X)$ let us say that g* **surpasses** *f and write $f \prec g$ if there is a real number $b > 0$ such that $f < bg$.*

7.6. THEOREM. *A Lindelöf space has CNP if and only if whenever f_1, f_2, $\cdots$ is a sequence of functions in $C(X)$, there is a $g \in C(X)$ that surpasses them all.*

PROOF. $\Leftarrow$: Every f_n will be bounded on any neighbourhood on which g is bounded.
$\Rightarrow$: Let f_1, f_2, ... be a sequence. We may assume, without loss of generality, that each $f_n > 1$. Using the SBP and the Lindelöf property, there is a countable cover U_1, U_2, ... of X such that for all n, $m \in \mathbf{N}$ each f_n is bounded on each U_m. Since a Lindelöf space is paracompact, there is a partition of unity $\{t_n\}$ subordinate to the cover. In fact, we may refine the cover and suppose that $U_n = \mathrm{coz}(t_n)$ (see [Kelley (1955), 5W and 5Y]). Let b_n be the sup of f_n on $U_1 \cup U_2 \cup \cdots \cup U_{n-1}$. Define $h_n = f_1 + f_2 + \cdots + f_n$ and $g = \sum_{n\in\mathbf{N}} h_n t_n$. The local finiteness guarantees that this sum is actually finite in a neighbourhood of each point, so continuity is clear.

We next claim that $x \in \mathrm{coz}(t_m)$ implies that $f_n(x) \leq b_n h_m(x)$ for all n and m. In fact, if $m < n$, then $f(x) \leq b_n \leq b_n h_m$ on $U_1 \cup U_2 \cup \cdots U_{n-1} \supseteq U_m = \mathrm{coz}(t_m)$. If $m \geq n$, then f_n is one of the summands of h_m so that $f_n \leq h_m \leq b_m h_m$. Note that the fact that each $f_n \geq 1$ everywhere implies the same for every h_n and b_n.

We can now finish the proof. Given a point $x \in X$, let $N(x)$ denote the finite set of indices n for which $t_n(x) \neq 0$. Then for all m,

$$b_n g(x) = \sum_{n\in\mathbf{N}} b_n h_n(x) t_n(x) = \sum_{n\in N(x)} b_n h_n(x) t_n(x)$$

$$\geq \sum_{n\in N(x)} f_m(x) t_n(x) = \sum_{n\in\mathbf{N}} f_m(x) t_n(x) = f_m(x)$$

7.7. Example. *Here is a nice application of Theorem 7.4. Say that a space satisfies the* **open refinement condition or ORC** *if the finite union closure of every ample G_δ cover has an open refinement. We explored this condition in some detail in [Barr, et al. (2007a)]. All P-spaces and all locally compact spaces satisfy the ORC and the class of spaces satisfying it is closed under finite products, closed subspaces and perfect images and preimages.*

7.8. Theorem. *For Lindelöf spaces, ORC implies CNP.*

Proof. Let X be Lindelöf and satisfy the ORC and let f_1, $f_2, \ldots$ be a sequence of functions in $C(X)$. For each compact set $A \subseteq X$ and each $n \in \mathbf{N}$, let $b_n(A) = \sup_{x \in A} |f_n(x)|$. Let $U_n(A) = \{x \in X \mid |f_n(x)| < b_n(A) + 1\}$ and $U(A) = \bigcap_{n \in \mathbf{N}} U_n(A)$. Then $U(A)$ is a G_δ containing A. The cover by the set of $U(A)$, taken over all the compact subsets of X, is an ample G_δ cover. Each f_n is bounded on each $U(A)$ and hence is bounded on the union of any finite number of them. Therefore each f_n is bounded on each set in an open refinement of $\{U(A)\}$. ∎

8. Outgrowths

The **outgrowth** of a Tychonoff space X is the space $\beta X - X$. In this section, we explore some of the ways a space and its outgrowth influence each other. By a **reduced outgrowth** of a Tychonoff space, we mean any space of the form $K - X$ where K is a compactification of X. Obviously a reduced outgrowth is a quotient of the outgrowth, but not every quotient is a reduced outgrowth.

8.1. Lemma. *Let K be a compactification of X. Then $K - X$ is countably compact if and only if $\beta X - X$ is.*

Proof. Let $\theta : \beta X \longrightarrow K$ be the canonical quotient map. If $\beta X - X$ is countably compact, then its image, $K - X$ clearly is too. To go the other way, we have from [Kelley (1955), 5E(a)] that if $\beta X - X$ is not countably compact, there is a sequence $S = \{p_1, p_2, p_3, \ldots\}$ that has no cluster point in $\beta X - X$. But then every cluster point in βX of the sequence lies in X. Either S has an infinite subsequence all of whose images in K are the same or S has an infinite subsequence all of whose images in K are distinct. We can suppose, without loss of generality that either $\theta(S)$ is a single point or that $\theta|S$ injective. The first case contradicts the fact that S is disjoint from X but has a cluster point in X . In the second case, $\theta(S)$ must have a cluster point in $K - X$. Let $\mathcal{V}$ be a non-principal (i.e, non-constant) ultrafilter on $\theta(S)$ which converges to a point of $K - X$. Then $\mathcal{V} = \theta(\mathcal{U})$ where $\mathcal{U}$ is a non-principal ultrafilter on S, which must converge to a point in X, leading to a contradiction. ∎

The proofs of the following are exercises.

8.2. Theorem. *Of the following conditions on a space X and a compactification K,*

1. *X is a P-set in K;*

2. *The closure in $K - X$ of every σ-compact subset of $K - X$ is compact;*

3. *The closure in $K - X$ of every countable subset of $K - X$ is compact;*

4. *$K - X$ is countably compact;*

5. *$K - X$ is pseudocompact.*

we have $1 \Leftrightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$.

8.3. Theorem. *If Y is a reduced outgrowth of a Lindelöf (respectively Alster) space, then any outgrowth of Y is Lindelöf (respectively Alster). Conversely, if some reduced outgrowth of Y is Lindelöf (respectively Alster), then Y is a reduced outgrowth of a Lindelöf (respectively Alster) space.*

Proof. Suppose X is a Lindelöf space and K is a compactification of X with $Y = K - X$. Then $L = \mathrm{cl}_K(Y)$ is a compactification of Y and $L - Y = L \cap X$ is closed in X and hence Lindelöf. Since $L - Y$ is a perfect image of $\beta Y - Y$ (see 2.1), it follows that $\beta Y - Y$ is Lindelöf (2.3) and hence the same is true for any compactification of Y. The Alster case goes the same way.

For the converse, suppose L is a compactification of Y with $L - Y$ Lindelöf. Let $\mathbf{N}^*$ be the one point compactification of $\mathbf{N}$ and $K = \mathbf{N}^* \times L$. Embed Y as $\{\infty\} \times Y$ and let $X = K - Y$. Clearly K is a compactification of X with reduced outgrowth Y. Finally, $X = (\mathbf{N} \times L) \cup \{\infty\} \times (L - Y)$ is the union of countably many Lindelöf spaces and is therefore Lindelöf. The argument with Alster's condition is similar. ∎

8.4. Theorem. *A space with a locally compact reduced outgrowth is the union of a compact subset and a locally compact subset.*

Proof. Let $X \subseteq K$ be a compactification such that $Y = K - X$ is locally compact. Let $L = \mathrm{cl}_K(Y)$. Since Y is locally compact it is open in L and hence $L - Y$ is a compact subset of X. If $p \notin L - Y$, then p has a K-neighbourhood that does not meet Y and a K-closed K-neighbourhood inside it. Such a neighbourhood is a compact K-neighbourhood of p inside X. ∎

8.5. Theorem. *A locally Alster reduced outgrowth of a Lindelöf CNP space is locally compact.*

Proof. Suppose that X is a Lindelöf CNP space, K is a compactification of X and $Y = K - X$. Assume that each $p \in Y$ has an Alster neighbourhood. Let F denote the family of all $f \in C(K)$ that vanish nowhere on X. Then $\{\mathbf{Z}[f] \mid f \in F\}$ is readily seen to be an ample G_δ cover of Y, since for each compact set $A \subseteq Y$ there is an $f \in F$ that vanishes on A (Theorem 2.10). If $p \in Y$, there is a neighbourhood U of p that is covered by a countable family of $Z[f]$. This means that $U \subseteq \bigcup \mathbf{Z}[f_n]$. But then $\bigcap \operatorname{coz}(f_n)$ is a G_δ that contains X and, by the CNP, there is an open $V \subseteq K$ such that $X \subseteq V \subseteq \bigcap \operatorname{coz}(f_n)$ and then $K - V \supseteq U$ is a compact neighbourhood of p. ∎

9. SCZ spaces

We will say that a space satisfies the *SCZ* condition if every σ-compact subset is contained in a compact zeroset. This can be usefully broken into two separate conditions:

SCZ-1. The closure of any σ-compact subset is compact;

SCZ-2. Every compact subset is contained in a compact zeroset.

The following claim is immediate.

9.1. Proposition. *A reduced outgrowth of a space satisfies SZC-1 if and only if the space satisfies the CNP.* ∎

9.2. Proposition. *A reduced outgrowth of a Lindelöf space satisfies SCZ-2.*

Proof. If X is Lindelöf and K is a compactification, let $Y = K - X$. If A is a compact set in Y, then $K - A$ is an open set that contains X. Since X is Lindelöf, there is a cozeroset $U \subseteq K$ such that $X \subseteq U \subseteq K - A$ (Theorem 2.10). Then $K - U$ is a zeroset, hence compact, in K and evidently $A \subseteq K - U \subseteq Y$. ∎

9.3. Proposition. *A reduced outgrowth of a space that satisfies SCZ-1 satisfies the CNP.*

Proof. Suppose X satisfies SCZ-1 and K is a compactification of X. Let $Y = K - X$ and $L = \operatorname{cl}_K(Y)$. Then L is a compactification of Y and hence $L - Y$ is a reduced outgrowth of Y. But $L - Y$ is a closed subspace of X and hence also satisfies SCZ-1. Let $U = \bigcap_{n \in \mathbf{N}} U_n$ (each U_n open in L), be a G_δ set in L that contains Y. It follows that for each n, $L - U_n$ is a compact subset of $L - Y$. According to SCZ-1, there is a compact set $A \subseteq L - Y$ with $L - U_n \subseteq A$ for all $n \in \mathbf{N}$. But then $L - A$ is an open set with $Y \subseteq L - A \subseteq U$. ∎

9.4. Proposition. *A space that satisfies SCZ-1 is pseudocompact.*

Proof. Suppose Y satisfies SCZ-1. If f were an unbounded function in $C(Y)$ we could choose points $p_1, p_2, \ldots, p_n, \ldots$ such that $|f(p_n)| > n$. The set $\{p_1, p_2, \ldots\}$ is discrete and not compact, but its closure is compact. If p is any point in its frontier, it is clear that $f(p)$ cannot be defined. ∎

9.5. Proposition. *A reduced outgrowth of any space that satisfies SCZ is Lindelöf.*

Proof. Let X satisfy SCZ. We will begin by showing that $Y = \beta X - X$ is Lindelöf. By Smirnov's Theorem (see 2.10), it will suffice to show that any open subset of $\mathrm{cl}_{\beta X}(Y)$ that contains Y contains a cozeroset containing Y. It will be sufficient to show the same with βX replacing $\mathrm{cl}_{\beta X}(Y)$. If U is an open subset of βX containing Y, $\beta X - U$ is closed in βX and hence compact and $\beta X - U \subseteq X$. Then there is a function $f : X \longrightarrow [0, 1]$ such that $\mathbf{Z}[f]$ is compact and $\beta X - U \subseteq \mathbf{Z}[f]$. The function f extends to a function $\widehat{f} : \beta X \longrightarrow [0, 1]$. We claim that $\mathbf{Z}[\widehat{f}]$ does not meet Y. For suppose that $p \in Y$ with $\widehat{f}(p) = 0$. Since $Z[f]$ is compact and therefore closed in βX, there is a function $g : \beta X \longrightarrow [0, 1]$ such that $g(p) = 0$ and $g(\mathbf{Z}[f]) = 1$. Then $\widehat{f} + g$ vanishes nowhere on X since $g = 1$ wherever $f = 0$. The previous proposition implies that $1/(\widehat{f} + g)$ is bounded on X and hence bounded on βX. Thus $\widehat{f} + g$ cannot vanish anywhere on Y, in particular at p. It follows that $\mathbf{Z}[\widehat{f}]$ is a compact zeroset in βX that does not meet Y and then $Y \subseteq \beta X - \mathbf{Z}[\widehat{f}] \subseteq U$.

In the general case of a reduced outgrowth, any space of the form $K - X$, for a compactification K of X, is a quotient of $\beta X - X$. ∎

In the next two propositions, X denotes a space that satisfies the SCZ, K is a compactification of X, $Y = K - X$, and $L = \mathbf{N}^* \times K$. Embed K into L as $\{\infty\} \times K$ and similarly embed X and Y into the infinity slice. Let $Z = (\mathbf{N} \times K) \cup (\{\infty\} \times Y) = L - (\{\infty\} \times X)$.

9.6. Proposition. *The space Z is Lindelöf.*

Proof. The Since X satisfies the SZC, Y is Lindelöf by the preceding proposition. Obviously $\mathbf{N} \times K$ is Lindelöf and hence so is Z. ∎

9.7. Proposition. *If Y satisfies the CNP, so does Z.*

Proof. Suppose $Z \subseteq U = \bigcap U_n$ with each U_n open in L. Then $Y \subseteq \bigcap(K \cap U_n)$ and each $K \cap U_n$ is open in K. Since Y satisfies the CNP, there is an open $V \subseteq K$ such that $Y \subseteq V \subseteq U \cap K$. But then $\mathbf{N}^* \times V$ and $\mathbf{N} \times K$ are open in L and

$$Z \subseteq (\mathbf{N}^* \times V) \cup (\mathbf{N} \times K) \subseteq U$$ ∎

9.8. Theorem. *A reduced outgrowth of a Lindelöf CNP space satisfies SCZ; a space that satisfies the SCZ is the reduced outgrowth of a Lindelöf CNP space.* ∎

Proof. Propositions 9.1 and 9.2 give one direction. The previous proposition gives the converse. ∎

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