On productively Lindelöf spaces

Michael Barr *
Department of Mathematics and Statistics
McGill University, Montreal, QC, H3A 2K6

John F. Kennison
Department of Mathematics and Computer Science
Clark University, Worcester, MA 01610

R. Raphael
Department of Mathematics and Statistics
Concordia University, Montreal, QC, H4B 1R6

December 1, 2006

Abstract. We study conditions on a topological space that guarantee that its product with every Lindelöf space is Lindelöf. The main tool is a condition discovered by K. Alster and we call spaces satisfying his condition Alster spaces. We also study some variations on scattered spaces that are relevant for this question.

1 Introduction. It is well known that a product of two Lindelöf spaces need not be Lindelöf. On the other hand, many spaces are known whose product with every Lindelöf space is Lindelöf. Let us call such a space **productively Lindelöf**.

K. Alster, [Alster (1988)], discovered (and we rediscovered, [Barr, Kennison, & Raphael (2006), Section 4] and called *amply Lindelöf*) a property which we will here call **Alster's condition** that is sufficient—and possibly necessary—for a space to be productively Lindelöf . Our formulation of Alster's condition follows. It looks rather different from Alster's but the two are readily shown to be equivalent.

Definition 1. *A space satisfies Alster's condition if every cover by G_δ sets that covers each compact set finitely contains a countable subcover. A space that satisfies this condition will be called an* **Alster space**.

The point about covering each compact set finitely is crucial. In the space of real numbers, every point is a G_δ but the cover by points has no proper subcover. But the reals are σ-compact and it is obvious that every σ-compact space is an Alster space. It is a trivial observation that if a space has the property that every compact set is a G_δ (this is the case in any metric space), then it is Alster if and only if it is σ-compact.

Alster proves, assuming CH, that a space of weight of at most $\aleph_1$ is productively Lindelöf if and only if it is Alster. However, it is quite evident in his paper that the "if" direction uses neither CH nor the weight condition; thus he showed that his condition implies productively Lindelöf . Neither he nor any of us is aware of any productively Lindelöf space that is *not* Alster, no matter the weight or set theory.

1991 *Mathematics Subject Classification.* 54D20, 54B10, 54G12.

Key words and phrases. productively Lindelöf, Alster space.

*The first and third authors would like to thank NSERC of Canada for their support of this research. We would all like to thank McGill and Concordia Universities for partial support of the middle author's visits to Montreal.

A somewhat different proof that Alster implies productively Lindelöf is found in [Barr, Kennison, & Raphael (2006), Theorem 4.5] where it is also shown that the product of two Alster spaces is Alster.

Oddly, despite interest in the question of productively Lindelöf, Alster's paper does not seem to be widely known. We have found only two citations, one by Alster himself and one in a paper that is widely unavailable (and we have not been able to see).

The paper [Telgarsky, 1971] studies C-scattered spaces in some detail (see Section 5 for the definition). One result we will be showing is that Lindelöf C-scattered spaces (and some more general ones) satisfy Alster's condition and therefore are productively Lindelöf.

2 Definitions and basic properties. All spaces considered here are completely regular and Hausdorff. We denote by $C(X)$ the ring of continuous real-valued functions on the space X.

We will be dealing with covers by G_δ sets. Since a finite union of G_δ sets is again a G_δ set, we can, and often will, suppose that the covers are closed under finite unions.

We recall that a continuous map $\theta : X \to Y$ is called **perfect** if it closed and if $\theta^{-1}(y)$ is compact for all $y \in Y$. It can be shown that whenever $B \subseteq Y$ is compact, $\theta^{-1}(B)$ is also compact. It is not always assumed that a perfect map is continuous, but we will suppose that it is. The inclusion map of a subspace is perfect if and only if the subspace is closed.

We also recall from [Barr, Kennison, & Raphael (2006), 2.2] that any $\theta : X \to Y$ induces three maps on subsets, the direct image also denoted $\theta : \mathcal{P}(X) \to \mathcal{P}(Y)$, the inverse image $\theta^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ and the universal image $\theta_\# : \mathcal{P}(X) \to \mathcal{P}(Y)$. These are characterized by the fact that if $A \subseteq X$ and $B \subseteq Y$, then $\theta(A) \subseteq B$ if and only if $A \subseteq \theta^{-1}(B)$ and $\theta^{-1}(B) \subseteq A$ if and only if $B \subseteq \theta_\#(A)$. Since $\theta_\#(A) = Y - \theta(X - A)$, we see that when θ is closed, $\theta_\#$ takes open sets to open sets.

We also recall that if X is a space, a point $p \in X$ is called a **P-point** if for any $f \in C(X)$, the set $\{q \in X \mid f(q) = f(p)\}$ is a neighbourhood of p. A **P-space** is a space in which every point is a P-point. It is immediate that P-spaces are characterized by the fact that G_δ sets are open.

A not-necessarily-open cover of a space is called **ample** if it covers every compact set finitely. We will say that a point $p \in X$ satisfies the **open refinement condition (ORC)** if every ample G_δ cover that is closed under finite union contains a neighbourhood of p. If p is a P-point, it satisfies the ORC because one element of the cover contains p and a G_δ that contains p is a neighbourhood of p. If p has a compact neighbourhood A then some member of the cover contains A and hence is a neighbourhood. Thus this condition is a common generalization of being a P-point and having a compact neighbourhood. We will say that a space **satisfies the ORC** or that it is **an ORC space** if every point satisfies the ORC. This is a common generalization of P-spaces and locally compact spaces. We will see that the class of ORC spaces is closed under finite products and closed subspaces and hence gives a much broader class than simply the union of the P-spaces and the locally compact spaces.

One of the main results of this paper is that Lindelöf ORC-scattered spaces (defined in Section 5) are Alster (see 25).

Theorem 2. *Of the following properties on a space:*

1. *discrete*
2. *P-space;*
3. *locally compact;*

4. *ORC;*

5. *Alster;*

6. *productively Lindelöf.*

1 implies 2 and 3, each of which implies 4. Condition 4 for Lindelöf spaces implies 5 and 5 implies 6.

Proof. We have already discussed the facts that 2 and 3 imply 4. It is clear that, for Lindelöf spaces, 4 implies 5 and Alster showed that 5 implies 6 (see also [Barr, Kennison, & Raphael (2006), Theorem 4.5]).

The space of rational numbers gives an example that is Alster but not ORC, since the cover by compact sets is an ample G_δ cover and every compact subset of the space is nowhere dense.

3 Permanence properties. A G_δ *cover* of a space is a cover by G_δ sets. We will say that a cover $\mathcal{U}$ of a space is *closed under finite unions* if $U, V \in \mathcal{U}$ implies that $U \cup V \in \mathcal{U}$. We will say that $\mathcal{U}$ is a G_δ *ideal* if it is closed under finite unions and if, whenever V is a G_δ subset of U and $U \in \mathcal{U}$, then also $V \in \mathcal{U}$. It is easy to see that a union of finitely many G_δ sets is a G_δ set so that the finite union closure and the G_δ ideal closure of a G_δ cover is also a G_δ cover. Moreover, the original cover has a countable refinement if and only if each of the finite union closure and the G_δ ideal closure does.

By substituting open for G_δ in the preceding paragraph, we can say that an open cover is finite union closed or an open ideal.

Theorem 3. *The product of two ORC spaces is an ORC space.*

We use several lemmas.

Lemma 4. *Let X and Y be spaces and $\mathcal{W}$ be an ample G_δ cover of $X \times Y$. Then for any compact sets $A \subseteq X$ and $B \subseteq Y$, and any $W \in \mathcal{W}$ such that $A \times B \subseteq W$, there are G_δ sets $U \subseteq X$ and $V \subseteq Y$ with $A \times B \subseteq U \times V \subseteq W$.*

Proof. Let $W = \bigcap_{n \in \mathbf{N}} W_n$ with each W_n open. According to [Kelley (1955), Theorem 5.12] there are, for each $n \in \mathbf{N}$ open sets $U_n \subseteq X$ and $V_n \in Y$ with $A \times B \subseteq U_n \times V_n \subseteq W_n$. If we let $U = \bigcap U_n$ and $V = \bigcap V_n$, then U and V are G_δ sets. □

Lemma 5. *Let X and Y be spaces with X being ORC. Let $\mathcal{W}$ be an ample G_δ ideal cover of $X \times Y$. Then for any compact set $B \subseteq Y$, there is an open ideal cover $\mathcal{U}(B)$ of X with the property that whenever $U \in \mathcal{U}(B)$ there is a G_δ set $V \subseteq Y$ with $B \subseteq V$ and $U \times V \in \mathcal{W}$.*

Proof. Let $\mathcal{U}'(B)$ denote the set of all G_δ sets $U \subseteq X$ for which there is a G_δ set $V \supseteq B$ with $U \times V \in \mathcal{W}$. If $U_1, U_2 \in \mathcal{U}'(B)$ there are G_δ sets V_1, V_2 containing B such that $U_i \times V_i \in \mathcal{W}$ for $i = 1, 2$. But then $B \subseteq V_1 \cap V_2$ and

$$(U_1 \cup U_2) \times (V_1 \cap V_2) \subseteq (U_1 \times V_1) \cup (U_2 \times V_2) \in \mathcal{W}$$

It is trivial to see that $\mathcal{U}'(B)$ is closed under G_δ subsets and hence is a G_δ ideal. Finally, if $A \subseteq X$ is compact, the ampleness of $\mathcal{W}$ implies that there is some $W \in \mathcal{W}$ that contains $A \times B$ and the preceding lemma gives G_δ sets U and V with $A \times B \subseteq U \times V \subseteq W$. Thus $U \in \mathcal{U}'(B)$. This shows that $\mathcal{U}'(B)$ is an ample G_δ ideal cover of X. The set $\mathcal{U}(B)$ of open sets in $\mathcal{U}'(B)$ is the required open ideal cover. □

Lemma 6. *If, in addition to the hypotheses of the preceding lemma, Y is also an ORC space, then for each compact $A \in X$, there is an open cover $\mathcal{V}(A)$ of Y such that for each $V \in \mathcal{V}(A)$ there is an open set $U \subseteq X$ such that $A \subseteq U$ and $U \times V \in \mathcal{W}$.*

Proof. Let $\mathcal{V}'(A)$ denote the set of all G_δ sets $V \subseteq Y$ for which there is an open set $U \subseteq X$ that contains A and for which $U \times V \in \mathcal{W}$. To show that $\mathcal{V}'(A)$ is ample, let $B \in Y$ be compact. According to the preceding lemma, there is an open ideal cover $\mathcal{U}(B)$ of X with the property that for all $U \in \mathcal{U}(B)$ there is a G_δ set $V \subseteq Y$ such that $B \subseteq V$ and $U \times V \in \mathcal{W}$. Since $\mathcal{U}(B)$ is an open ideal cover, there is a $U \in \mathcal{U}(B)$ with $A \subseteq U$ and this shows that $V \in \mathcal{V}'(A)$. The fact that $\mathcal{V}'(A)$ is an ideal cover follows exactly as in the preceding lemma. The set $\mathcal{V}(A)$ of open sets in $\mathcal{V}'(A)$ is the required cover. □

Proof of Theorem 3. Let $\mathcal{W}$ be an ample G_δ ideal cover of $X \times Y$. Given any point $(x, y) \in X \times Y$, let V be an element of $\mathcal{V}(\{x\})$ that contains y. By definition, there is an open set $U \subseteq X$ with $x \in U$ such that $(x, y) \in U \times V \in \mathcal{W}$. Thus the open sets in $\mathcal{W}$ cover $X \times Y$.□

The table of permanence properties below has one row labeled "Local". A property of spaces is **local** provided a space has the property if and only if every neighbourhood of every point contains a neighbourhood that has the property. The properties here are all closed under formation of closed subspaces and the closed subspaces are a neighbourhood base, so that to verify that a property is local, we need assume only that each point has some neighbourhood with that property.

Theorem 7. *The following table expresses the permanence of these properties. In this table, D means discrete, P means P-space, LC means locally compact, A means Alster, and PL means productively Lindelöf. A + sign indicates the property is preserved, while a − sign means it is not necessarily preserved.*

	D	*P*	*LC*	*ORC*	*A*	*PL*
Finite products	+	+	+	+	+	+
Perfect preimage	*	*	+	+	+	+
Closed subspaces	+	+	+	+	+	+
Continuous image	−	−	−	−	+	+
Local	+	+	+	+	**	**
Quotient	+	+	−	−	+	+
Open image	+	+	+	+	+	+
Perfect image	+	+	+	+	+	+

** Preserved, provided the inverse image of each point is finite.*
*** Yes, provided the total space is Lindelöf.*

Proof. We will verify only the positive properties here as the negative ones are not used in this paper and examples are mostly easy. See 6.3 to see that local compactness and satisfaction of ORC do not pass to quotients. Certain properties follow from others and will not be mentioned explicitly: a closed subspace is a perfect preimage and both open images and closed images are quotients while quotients are continuous images. We will take each class of spaces in turn.

Discrete: Obvious.

P-space: See [Gillman & Jerison (1960), 4K] for products. Closure under subspaces and quotient mappings is obvious. It is known that a perfect preimage of a P-space need not be a P-space. We will prove it here under the additional hypothesis that the inverse image of each point is a singleton or doubleton. Any finite-to-one map will work, but the notation gets ugly. So let $\theta : Y \to X$ be such a map with X a P-space. Let $y \in Y$ and $f : Y \to [0,1]$ be continuous with $f(y) = 0$. Suppose that $\theta(y') = \theta(y)$ and $f(y') = 1$. The case that there is no such y' or that $f(y') = 0$ is easier and we omit it. For each pair of positive integers m and n, the set

$$U_{mn} = \{p \mid f(p) < 1/m\} \cup \{q \mid 1 - f(q) < 1/n\}$$

is an open neighbourhood of $\{y, y'\}$ and hence $\theta_{\#}(U_{mn})$ is an open neighbourhood of $\theta_{\#}(y, y') = \theta(y)$. Since X is a P-space, $\bigcap \theta_{\#}(U_{mn}) = \theta_{\#}(\bigcap U_{mn})$ is a neighbourhood of $\theta(y)$ and hence $\theta^{-1}(\theta_{\#}(\bigcap U_{mn})) \subseteq \bigcap U_{mn}$ is a neighbourhood of $\{y, y'\}$. But $\bigcap U_{mn} = f^{-1}(0) \cup f^{-1}(1)$ and the only way it can be a neighbourhood of $\{y, y'\}$ is for the first component to be a neighbourhood of y and the second a neighbourhood of y'. Now suppose that every point has a neighbourhood that is a P-space. Since every subspace of a P-space is a P-space, we can assume that every point has an open P-space neighbourhood. A G_δ set will meet every such neighbourhood in an open set and hence is a union of open sets.

Locally compact: It is well-known that a finite product of locally compact spaces is locally compact. It is shown in [Engelking, (1989), p. 189] that local compactness is closed under perfect image and preimage. It is obvious that the open image of a locally compact space is locally compact. Local compactness is the quintessential local property.

ORC: The closure under products is Theorem 3.

If $\theta : Y \to X$ is perfect and X is ORC, let $\mathcal{V}$ be an ample G_δ cover of Y. Assume it is closed under finite unions. For any $y \in Y$, $\theta^{-1}(\theta(y))$ is compact and hence contained in some $V \in \mathcal{V}$. It follows that $\theta(y) \in \theta_{\#}(\theta^{-1}(\theta(y))) \subseteq \theta_{\#}(V)$ so that $\theta_{\#}(\mathcal{V})$ is a G_δ cover of X. A similar argument shows it is ample. The finite sum closure has an open cover refinement and the inverse image of that refinement refines $\mathcal{V}$.

Assume that each $x \in X$ has an ORC neighbourhood $U(x)$. Let $\mathcal{V}$ be an ample G_δ cover of X and assume that $\mathcal{V}$ is a G_δ ideal cover. Then $\mathcal{V}$ restricted to $U(x)$ has an open subcover. In particular, there is some set $V \in \mathcal{V}$ that contains x and is open relative to $U(x)$. But then $x \in V \cap \operatorname{int}(U(x)) \in \mathcal{V}$, and $V \cap \operatorname{int}(U(x))$ is open in X.

If $\theta : X \to Y$ is open and X is ORC, let $\mathcal{V}$ be an ample G_δ cover of Y, which we will suppose closed under finite union. Then $\theta^{-1}(\mathcal{V}) = \{\theta^{-1}(V) \mid V \in \mathcal{V}\}$ is an ample G_δ cover of X and also closed under finite union. Thus there is an open cover $\mathcal{U}$ such that for all $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ with $U \subseteq \theta^{-1}(V)$, which implies that $\theta(U) \subseteq V$. Moreover $\theta(U)$ is open by hypothesis and hence $\theta(\mathcal{U}) = \{\theta(U) \mid U \in \mathcal{U}\}$ is an open refinement of $\mathcal{V}$.

Let $\theta : X \to Y$ be perfect and assume that X is ORC. If $\mathcal{V}$ is an ample G_δ cover of Y, closed under finite unions, then $\theta^{-1}(\mathcal{V})$ is an ample G_δ cover of X closed under finite unions. It therefore has an open cover refinement $\mathcal{U}$, which may be assumed to be closed under finite unions. Then each compact set in X, in particular, every set of the form $\theta^{-1}(y)$, is contained in a single set of $U \in \mathcal{U}$. This implies that $y \in \theta_{\#}(U)$. Thus $\theta_{\#}(\mathcal{U})$ is an open cover refinement of $\mathcal{V}$.

Alster: For finite products, see [Barr, Kennison, & Raphael (2006), Theorem 4.5]. Suppose $\theta : Y \to X$ is perfect and X is Alster. Let $\mathcal{V}$ be an ample G_δ cover of Y. If $p \in Y$, $\theta^{-1}(\theta(p))$ is compact and hence contained in some $V \in \mathcal{V}$ so that $\theta(p) = \theta_{\#}(\theta^{-1}(\theta(p))) \in \theta_{\#}(V)$ and thus $\theta_{\#}(\mathcal{V})$ is a cover of X. Since $\theta_{\#}$ preserves open sets and meets, $\theta_\#(\mathcal{V})$ is a G_δ cover. If $A \in X$ is compact, $\theta^{-1}(A)$ is compact and therefore contained in some $V \in \mathcal{V}$, whence $A = \theta_\#(\theta^{-1}(A)) \subseteq \theta_\#(V)$. Thus $\theta_\#(\mathcal{V})$ is an ample G_δ cover of X and has a countable subcover $\mathcal{U}$. If $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ such that $U \subseteq \theta_\#(V)$, from which we conclude that $\theta^{-1}(U) \subseteq \theta^{-1}(\theta_\#(V)) \subseteq V$. Suppose $\theta : X \to Y$ is a continuous surjection and X is Alster. Let $\mathcal{V}$ be an ample G_δ cover of Y. Then $\theta^{-1}(\mathcal{V})$ is a G_δ cover of X. Since the image of a compact space is compact, it is clear that $\theta^{-1}(\mathcal{V})$ is ample. There is a countable subcover $\{\theta^{-1}(V_1), \theta^{-1}(V_2), \ldots,\}$ and the corresponding $\{V_1, V_2, \ldots\}$ is a countable subcover of $\mathcal{V}$. If X is Lindelöf and every point has an Alster neighbourhood, there is a countable family of them whose interiors cover X. The sum of that countable family of subspaces is Alster and X is a continuous image of the sum.

Productively Lindelöf: Closure under finite products follows from the definition. Most of the remaining properties follow from the corresponding properties of Lindelöf spaces. The one that does not is localness and the proof works exactly the same as for Alster. □

4 Derived spaces. If S is a property of topological spaces, let us call a space that has that property an S**-space**. An S-subspace is a subspace having property S; if it is a neighbourhood of some point, we will call it an S-neighbourhood of that point. We have in mind mainly the following four properties:

D: being discrete;

P: being a P-space;

C: being compact;

ORC: being ORC.

Hypothesis 8. *We will suppose that S satisfies the following conditions:*

1. *A closed subspace of an S-space is an S-space.*
2. *The union of two closed S-subspace is an S-subspace.*
3. *The product of two S-spaces is an S-space.*

Proposition 9. *The four examples D, P, C, and ORC satisfy these conditions.*

Proof. The first and third of these is either evident or follows from Theorem 7. As for the second, the union of two closed subspaces is a perfect image of their sum and it is obvious that these conditions are all preserved by sums. □

From here until Theorem 25, S stands for any property of spaces that satisfies Hypotheses 8.

Define

$$L_S(X) = \{p \in X \mid p \text{ has an } S\text{-neighbourhood}\}$$

and $D_S(X) = X - L_S(X)$. We define $D_S^\alpha(X)$ for any ordinal α inductively by $D_S^\alpha = D_S(D_S^\beta)$ when $\alpha = \beta + 1$ and, if α is a limit ordinal, then $D_S^\alpha = \bigcap_{\beta<\alpha} D_S^\beta$. Evidently, $D_S^\alpha(X)$ is closed in X for all α.

From now on, we will usually suppress the S and write $L(X)$ and $D(X)$ for $L_S(X)$ and $D_S(X)$, respectively.

Proposition 10. *If A is an open or closed subset of X, then $L(A) \supseteq A \cap L(X)$ and $D(A) \subseteq A \cap D(X)$. When A is open these inclusions are equalities.*

Proof. Suppose first that A is closed. If $p \in A$ and $U \subseteq X$ is an S-neighbourhood of p, then $A \cap U$ is closed in U and is therefore an S-neighbourhood of p in A and so $p \in L(A)$. We have

$$D(A) = A - L(A) \subseteq A - (A \cap L(X)) \subseteq A \cap (X - L(X)) = A \cap D(X)$$

When A is open, suppose $p \in A \cap L(X)$. Let U be an S-neighbourhood of p in X and let V be a closed neighbourhood of p inside A. Then $U \cap V$ is an S-neighbourhood of p so that $p \in L(A)$. Conversely, if $p \in L(A)$ then p has an S-neighbourhood inside A, but, A being open in X, this is also an S-neighbourhood in X. □

Proposition 11. *If A is a closed or open subset of X, then $D^\alpha(A) \subseteq A \cap D^\alpha(X)$ for all α; when A is open, the inclusion is an equality.*

Proof. First suppose that A is closed. If we suppose that $D^\beta(A) \subseteq D^\beta(X)$ then $D^\beta(A)$ is closed in A, which is closed in X and therefore $D^\beta(A)$ is closed in $D^\beta(X)$ so that

$$D^{\beta+1}(A) = D(D^\beta(A)) \subseteq D(D^\beta(X)) = D^{\beta+1}(X)$$

from which the conclusion is obvious. The same conclusion holds at limit ordinals by taking intersections.

Now let A be open. If we suppose that $D^\beta(A) = A \cap D^\beta(X)$, then since A is open, $A \cap D^\beta(X)$ is open in $D^\beta(X)$ so that

$$D^{\beta+1}(A) = D(D^\beta(A)) = D(A \cap D^\beta(X)) = A \cap D^\beta(X) \cap D^{\beta+1}(X) = A \cap D^{\beta+1}(X)$$

Again, the same conclusion holds at limit ordinals by taking intersections. □

Proposition 12. *If A and B are both open or both closed subsets of X, then $D(A \cup B) = D(A) \cup D(B)$.*

Proof. For open sets, we have from Proposition 11 that $D(A) = A \cap D(X)$ and $D(B) = B \cap D(X)$ so that $D(A) \cup D(B) = (A \cup B) \cap D(X) = D(A \cup B)$. For closed sets, we have from Proposition 11 that $D(A) \subseteq A \cap D(A \cup B)$ and $D(B) \subseteq B \cap D(A \cup B)$, so that $D(A) \cup D(B) \subseteq (A \cup B) \cap D(A \cup B) = D(A \cup B)$. For the reverse inequality we must show that

$$A \cup B - L(A \cup B) \subseteq (A - L(A)) \cup (B - L(B))$$

In other words, that if $p \in A \cup B$ and $p \notin L(A \cup B)$, then either $p \in A$ and $p \notin L(A)$ or $p \in B$ and $p \notin L(B)$.

If $p \in A - B$ and $p \in L(A)$, then p has an S-neighbourhood and, since B is closed, p has a closed neighbourhood disjoint from B. Their intersection is an S-neighbourhood disjoint from B, which is then an S-neighbourhood of p in $A \cup B$ so that $p \in L(A \cup B)$. If $p \in B - A$, we have the same argument. Finally we consider the case that $p \in A \cap B$ and $p \in L(A) \cap L(B)$. Then p has a closed S-neighbourhood $U \subseteq A$ and a closed S-neighbourhood $V \subseteq B$. Let U' and V' be $A \cup B$-neighbourhoods of p such that $U' \cap A = U$ and $V' \cap B = V$. Then $U' \cap V'$ is an $(A \cup B)$-neighbourhood of p and

$$U' \cap V' = (U' \cap V') \cap (A \cup B) = (U' \cap V' \cap A) \cup (U' \cap V' \cap B) = (U \cap V') \cup (U' \cap V) \subseteq U \cup V$$

so that $U \cup V$ is an S-neighbourhood of p in $(A \cup B)$. □

Corollary 13. *If A and B are both open or both closed subsets of X, then for any ordinal α, $D^\alpha(A \cup B) = D^\alpha(A) \cup D^\alpha(B)$.* □

Proposition 14 (Leibniz formula)**.** *For any spaces X and Y, $D(X \times Y) \subseteq (X \times D(Y)) \cup (D(X) \times Y)$.*

Proof. Since $L(X)$ and $L(Y)$ satisfy S, so does $L(X) \times L(Y) \subseteq X \times Y$ so that $L(X \times Y) \supseteq L(X) \times L(Y)$ from which the conclusion is clear. □

Corollary 15. *For all $n \in \mathbf{N}$, $D^n(X \times Y) \subseteq \bigcup_{i+j=n}(D^i(X) \times D^j(Y))$.* □

Proposition 16. *For all $n > 0$ in $\mathbf{N}$, $D^{2n-1}(X \times Y) \subseteq (X \times D^n(Y)) \cup (D^n(X) \times Y)$.*

Proof.

$$\begin{aligned} D^{2n-1}(X \times Y) &\subseteq \bigcup_{i=0}^{n-1}(D^i(X) \times D^{2n-1-i}Y) \cup \bigcup_{i=n}^{2n-1}(D^i(X) \times D^{2n-1-i}Y) \\ &\subseteq (X \times D^n(Y)) \cup (D^n(X) \times Y) \end{aligned}$$

□

Corollary 17. $D^\omega(X \times Y) \subseteq (X \times D^\omega(Y)) \cup (D^\omega(X) \times Y)$.

Proof.

$$\begin{aligned} D^\omega(X \times Y) &= \bigcap((X \times D^n(Y)) \cup (D^n(X) \times Y)) = \bigcap(X \times D^n(Y)) \cup \bigcap(D^n(X) \times Y) \\ &= (X \times D^\omega(Y)) \cup (D^\omega(X) \times Y) \end{aligned}$$

where the commutation of the meet and join is justified by the fact that the sequences of $D^n(X)$ and $D^n(Y)$ are descending. □

Theorem 18. *Assume the Hypothesis 8. Then for any limit ordinal α, we have*

$$D^\alpha(X \times Y) \subseteq (X \times D^\alpha(Y)) \cup (D^\alpha(X) \times Y)$$

Proof. Either $\alpha = \beta + \omega$ with β a limit ordinal, or $\alpha = \bigcup \beta$, the latter union over all the limit ordinals below α. In the first case, we can suppose by induction that the result is valid for β and then we see that $D^{\beta+2n-1}(X \times Y) \subseteq D^\beta(X) \times D^{\beta+n}(Y) \cup D^{\beta+n}(X) \times D^\beta(Y)$. Forming the meet over all n, we conclude the result for α. In the second case, we assume inductively that the conclusion is true for all $\beta < \alpha$ and form the meet over all such β. □

5 Scattered spaces.

If S is a property of topological spaces, we will say that a space X is **S-scattered** if for some ordinal α, $D^\alpha_S(X) = \emptyset$. The following is an immediate consequence of the preceding section.

Theorem 19. *An open or closed subspace of an S-scattered space, a union of two open or two closed S-scattered subspaces and a product of two S-scattered spaces, is S-scattered.* □

The following is an immediate consequence of Proposition 11:

Proposition 20. *A space X is S-scattered if and only if every non-empty closed subset $A \subseteq X$ contains an S-space whose A-interior is non-empty.*

Corollary 21. *A space that is S-scattered for $S = D$, P, or C is also ORC-scattered.*

Proposition 22. *Suppose $X = Y \cup Z$ and both Y and Z are S-scattered. If one of Y or Z is either open or closed in X, then X is also S-scattered.*

Proof. Suppose Y is open. From Proposition 11, we have that for all ordinals α, $D^\alpha(Y) = Y \cap D^\alpha(X)$. If α is chosen so that $D^\alpha(Y) = \emptyset$, we conclude that $D^\alpha(X) \subseteq Z$ and then the result follows since Z is scattered.

Now suppose that Y is closed. Then $X - Y$ is an open subset of X and therefore scattered, so the result follows from the first part applied to $(X - Y) \cup Y$. □

Corollary 23. *The union of finitely many S-scattered subspaces, each of which is either open or closed, is S-scattered.* □

One can show that if a Lindelöf space X contains an open subspace U for which U and $X - U$ are P-spaces, then X is Alster. This is a special case of the following (a space is δ-Lindelöf if it is Lindelöf when retopologized by taking the G_δ sets of the original topology as a neighbourhood base.)

Theorem 24 ([Henriksen *et. al.*, (to appear)])**.** *A Lindelöf P-scattered space is δ-Lindelöf.*

Using a transfinite induction argument similar to that of [Henriksen *et. al.*, (to appear)], we will prove:

Theorem 25. *A Lindelöf ORC-scattered space is Alster.*

It follows from Corollary 21 that this theorem will show that any Lindelöf D, P, C, or ORC-scattered space is Alster.

Proof. Suppose X is a space and $D^\alpha(X) = \emptyset$. We make the inductive hypothesis that, for any $\beta < \alpha$, any Lindelöf space Y for which $D^\beta(Y) = \emptyset$, is Alster. We first consider the case that α is a limit ordinal. In that case, $X = \bigcup_{\beta<\alpha}(X - D^\beta(X))$, which is a union of open sets. For any $\beta < \alpha$ and any element $x \in X - D^\beta(X)$, there is a closed neighbourhood $V(x)$ of x contained in $X - D^\beta(X)$. Since $V(x)$ is closed, it follows from Proposition 11 that $D^\beta(V(x)) \subseteq V(x) \cap D^\beta(X - D) = \emptyset$. It also follows that $V(x)$ is Lindelöf and hence, by the inductive hypothesis, that $V(x)$ is Alster. Thus every point of X has an Alster neighbourhood. Since X is Lindelöf, the localness of Alster (Theorem 7) implies that X is Alster.

Now suppose that $\alpha = \beta + 1$ is a successor. In that case, every element of $Y = D^\beta(X)$ has an ORC neighbourhood. We showed in Theorem 7 that being ORC is a local property, and hence Y is ORC. Let $\mathcal{U}$ be an ample G_δ cover of X. From [Barr, Kennison, & Raphael (2006), 4.8] we may suppose, without loss of generality, that $\mathcal{U}$ consists of zerosets. Since a finite union of zerosets is a zeroset, we can suppose that $\mathcal{U}$ is closed under finite unions. Then $\mathcal{U}|Y = \{U \cap Y \mid U \in \mathcal{U}\}$ has an open refinement by sets of the form $V \cap Y$, where V is open in X. This open refinement has a further refinement by cozerosets of X. Since Y is Lindelöf, there is a countable subfamily, $\{V_n \mid n \in \mathbf{N}\}$ of cozerosets of X such that $\{V_n \cap Y\}$ covers Y and refines $\mathcal{U}|Y$. For each $n \in \mathbf{N}$, let $U_n \in \mathcal{U}$ be a set for which $V_n \cap Y \subseteq U_n$. Now $X - \bigcup V_n$ is closed in X and thus, by Proposition 11

$$D^\beta\left(X - \bigcup V_n\right) \subseteq \left(X - \bigcup V_n\right) \cap D^\beta(X) \subseteq (X - Y) \cap Y = \emptyset$$

and the inductive hypothesis implies that $X - \bigcup V_n$ is countably covered by $\mathcal{U}$. Each set $V_n - U_n$ is the difference of a cozeroset and a zeroset, which is a cozeroset, hence an F_σ, and therefore Lindelöf. If $V_n - U_n = \bigcup_m A_{nm}$ with each A_{nm} closed, we have

$$D^\beta(A_{nm}) \subseteq A_{nm} \cap Y \subseteq (V_n - U_n) \cap Y \subseteq (V_n \cap Y) - U_n = \emptyset$$

so that the inductive hypothesis implies that A_{nm} is countably covered by $\mathcal{U}$ and then so is $\bigcup_n (V_n - U_n) = \bigcup_{n,m} A_{nm}$. Finally, $\bigcup_n U_n$ is countably covered by the U_n and so

$$X = \Big(X - \bigcup V_n\Big) \cup \Big(\bigcup (V_n - U_n)\Big) \cup \bigcup U_n$$

is countably covered. Thus X is Alster. □

Theorem 26. *When S is one of the classes D, P, C, or ORC, being S-scattered is preserved by perfect surjections. In case that S is C or ORC, being S-scattered is preserved by perfect preimage; in case that S is D or P, being S-scattered is preserved by preimage under perfect mappings in which the preimage of each point is finite.*

The proof will proceed by a series of lemmas. Note that all four of the classes are preserved by perfect image (which implies closure under finite unions of closed subobjects) and, subject to the proviso in the statement, perfect preimage (see Theorem 7).

Lemma 27. *Suppose $\theta : X \to Y$ is a perfect surjection. Then $\theta(D(X)) \supseteq D(Y)$.*

Proof. We must show that $y \notin L(Y)$ implies that there is some $x \in \theta^{-1}(y)$ such that $x \notin L(X)$. Equivalently, we must show that $x \in L(X)$ for all $x \in \theta^{-1}(y)$ implies $y \in L(Y)$. Suppose that for each $x \in \theta^{-1}(y)$ there is an S-neighbourhood $U(x)$ of x. We may suppose that each $U(x)$ is closed. Since $\theta^{-1}(y)$ is compact, there is finite set $x_1, \ldots, x_n \in \theta^{-1}(y)$ such that the interiors of $U(x_1), \ldots, U(x_n)$ cover $\theta^{-1}(y)$. Thus $\theta^{-1}(y) \subseteq U = \bigcup_{i=1}^n U(x_i)$ and so $\theta_\#(U)$ is a neighbourhood of y. By Theorem 7, $\theta(U)$ is an S-subspace of Y and also a neighbourhood of y since $\theta(U) \supseteq \theta_\#(U)$.

Lemma 28. *Suppose $\theta : X \to Y$ is a perfect surjection. Then for all ordinals α, $\theta(D^\alpha(X)) \supseteq D^\alpha(Y)$.*

Proof. If we make the inductive hypothesis that $\theta(D^\alpha(X)) \supseteq D^\alpha(Y)$, it follows that there is a perfect surjection $X_\alpha = \theta^{-1}(D^\alpha(Y)) \cap D^\alpha(X) \to D^\alpha(Y)$. Since $X_\alpha \subseteq D^\alpha(X)$, we have that $D(X_\alpha) \subseteq D^{\alpha+1}(X)$ so that $\theta(D^{\alpha+1}(X)) \supseteq \theta(D(X_\alpha)) \supseteq D^{\alpha+1}(Y)$. Now suppose that α is a limit ordinal and $\theta(D^\beta(X)) \supseteq D^\beta(Y)$ for all $\beta < \alpha$. We want to show that $\theta\left(\bigcap_{\beta<\alpha} D^\beta(X)\right) \supseteq D^\alpha(Y)$. For each $y \in D^\alpha(Y)$ and each $\beta < \alpha$ the set $\{x \in D^\beta(X) \mid \theta^{-1}(y)\}$ is a non-empty closed subset of the compact set $\theta^{-1}(y)$ and hence their meet over all $\beta < \alpha$ is non-empty. □

Corollary 29. *If $\theta : X \to Y$ is a perfect surjection and X is S-scattered, then so is Y.* □

In order to simplify the statements of the following results, we will say that a map is *S-perfect* if it is perfect and, in case $S =$ D or P, that the inverse image of each point is finite.

Lemma 30. *Suppose $\theta : X \to Y$ is S-perfect. Then $\theta(D(X)) \subseteq D(Y)$.*

Proof. We have

$$\begin{array}{rl} \theta(D(X)) \subseteq D(Y) & \text{if and only if} \\ \theta(X - L(X)) \subseteq Y - L(Y) & \text{if and only if} \\ Y - \theta_\#(L(X)) \subseteq Y - L(Y) & \text{if and only if} \\ L(Y) \subseteq \theta_\#(L(X)) & \text{if and only if} \\ \theta^{-1}(L(Y)) \subseteq L(X) & \end{array}$$

If $y \in L(Y)$, then y has an S-neighbourhood U. Then $\theta^{-1}(U)$ is a neighbourhood of each point of $\theta^{-1}(y)$ and, from Theorem 7, is an S-subset and hence each point of $\theta^{-1}(y)$ is in $L(U)$. □

Lemma 31. *Suppose $\theta : X \to Y$ is S-perfect. Then for all ordinals α, $\theta(D^\alpha(X)) \subseteq D^\alpha(Y)$.*

Proof. Assume by induction that $\theta(D^\alpha(X)) \subseteq D^\alpha(Y)$. Then

$$\theta(D^{\alpha+1}(X)) = \theta(D(D^\alpha(X))) \subseteq \theta(D(D^\alpha)(X)) \subseteq D(D^\alpha(Y)) = D^{\alpha+1}(Y)$$

If α is a limit ordinal and $\theta(D^\beta(X)) \subseteq D^\beta(Y)$ for all $\beta < \alpha$, then

$$\theta(D^\alpha(X)) = \theta\left(\bigcap_{\beta<\alpha} D^\beta(X)\right) \subseteq \bigcap \theta(D^\beta(X)) \subseteq \bigcap D^\beta(Y) = D^\alpha(X) \qquad \square$$

Corollary 32. *If $\theta : X \to Y$ is S-perfect and Y is S-scattered, so is X.* □

This finishes the proof of Theorem 26. As an application, we have:

Corollary 33. *Suppose $X = \bigcup_{i\in I} X_i$ is a locally finite union of closed S-scattered spaces. Then X is S-scattered.*

Proof. The canonical map from the categorical sum to the union is easily seen to be closed with the inverse images of points being finite. □

6 Examples. In this section, we give two examples to distinguish between some of these classes of spaces.

6.1 Example The first is an example of a space that is Alster, but does not satisfy the ORC. Let $p \in \beta X - X$ and $X = \mathbf{N} \cup \{p\}$ as a subspace of βX. Then X, being countable, is Alster. It is not ORC since each point is a G_δ and every compact set is finite, so the cover by finite sets is an ample G_δ cover that contains no neighbourhood of p. It is easy to see that the space is space is ORC-scattered, in fact $D^2_{\mathrm{ORC}}(X) = \emptyset$.

6.2 Scattering degrees. We define the **degree** of an S-scattered space X to be the smallest ordinal α for which $D^\alpha_S(X) = \emptyset$. The example above raises the question of whether we can find ORC-scattered spaces of arbitrary degree. We answer this question here. We claim that there exists a Lindelöf ORC-scattered space of degree α for any countable ordinal α. We further claim that if we drop the "Lindelöf", then there is an ORC-scattered space of any degree. We start by saying that the degree of a point $p \in X$, denoted $\deg_X(p)$ or $\deg(p)$ if X is clear, is the smallest ordinal γ for which $p \notin D^\gamma(X)$. We note that $\deg(p)$ can never be a limit ordinal. Clearly the degree of X is the supremum of $\{\deg(p) \mid p \in X\}$. Our claim about the degree of ORC-scattered spaces is a consequence of the constructions in the previous example and the proofs of the following lemma and theorem.

Lemma 34. *Let X be an ORC-scattered space. Let $p \in X$ be a point of degree α. For any point $q \in \beta\mathbf{N} - \mathbf{N}$, the space $Y = (X \times \mathbf{N}) \cup \{(p, q)\} \subseteq X \times \beta\mathbf{N}$ is ORC-scattered and (p, q) is a point of Y of degree $\alpha + 1$. If X is Lindelöf, so is Y.*

Proof. For each $n \in \mathbf{N}$ consider $X \times \{n\}$, which is a clopen subset of Y. By Proposition 11, $D^\gamma(X \times \{n\}) \subseteq D^\gamma(Y)$. It is obvious that $D^\gamma(X \times \{n\}) = D^\gamma(X) \times \{n\}$ so $(p, n) \in D^\gamma(Y)$ for all $n \in \mathbf{N}$. Since $D^\gamma(Y)$ is closed, we see that $(p, q) \in D^\gamma(Y)$ too. By the argument given in Example 6.1, applied to the closed subset $\{p\} \times (N \cup \{q\})$, we see that $(p, q) \notin D^{\gamma+1}(Y) = D^\alpha(Y)$. But (p, q) is an isolated point in $D^\alpha(Y)$ because $(X - D^\alpha(X) \times \mathbf{N} \cup \{(p, q)\}$ is a neighbourhood of (p, q) which meets $D^\alpha(Y)$ in just the point (p, q). The result follows. □

Theorem 35. *For each ordinal α, there is an ORC-scattered space of scattering degree α.*

Proof. Suppose by transfinite induction that this is true for all smaller ordinals. If α is a non-limit ordinal, the lemma gives the result. If α is a limit ordinal, choose, for each $\gamma < \alpha$, a space X_γ of scattering degree γ. Then $\sum X_\gamma$ has scattering degree α. □

6.3 Example Here is an example that illustrates several points. It is an example of the fact that local compactness is not preserved by quotients (in fact, see [Kelley (1955), Exercise 5N] for the example of which the present one is a slight modification); that satisfying ORC is not preserved under quotients; and another example that shows that an Alster space need not satisfy the ORC.

Let X denote the quotient of the space $[0,1]\times\mathbf{N}$ by the equivalence relation that identifies all the points $\{0\}\times\mathbf{N}$. We will call the point corresponding to the identified points p; all other points will be identified by their coordinates in the product.

There is a neighbourhood base at p that consists of all sets of the form $\{p\}\cup\bigcup_{n\in\mathbf{N}}((0,\epsilon_n)\times\{n\})$, where $0<\epsilon_n$ for each n. There is no uniformity in the choices of ϵ_n.

The space is completely regular. There is no problem separating a point different from p from a closed set not containing it. As for p, let $\{p\}\cup\bigcup((0,\epsilon_n)\times\{n\})$ be a basic neighbourhood of p. The function $f: X\to\mathbf{R}$ for which $f(p)=0$ and whose restriction $(0,1]\times\{n\}$ is given by $f(t,n)=t/\epsilon_n$ is continuous, vanishes at p, and ≥ 1 outside the neighbourhood and thus separates p from any closed set that does not meet the given neighbourhood.

We claim that every compact set is contained in a subset $\{p\}\cup\bigcup_{n\le m}((0,1]\times\{n\})$ for some $m\in\mathbf{N}$. Otherwise, let K be a compact set that is not contained in any such finite union. For each $n\in\mathbf{N}$, let t_n be the largest first coordinate of any element of $K\cap((0,1]\times\{n\})$, if any. If there are no such elements, let $t_n=1$. For each $m\in\mathbf{N}$, let $U_m=\{p\}\cup\bigcup_{n\le m}((0,1]\times\{n\})\cup\bigcup_{n>m}((0,t_n)\times\{n\})$. Then $\{U_n\}$ is an increasing sequence of open sets that covers K, but no single one does and hence K is not compact.

It follows that no neighbourhood of p has a compact closure so that X is not locally compact. We have just shown that the cover by the sets $\{p\}\cup((0,1]\times\{1,2,\ldots,m\})$, for $m\in\mathbf{N}$, is ample and it obviously contains no neighbourhood of p. These sets are G_δ since the mth set is

$$\bigcap_{n=1}^{\infty}(\{p\}\cup((0,1]\times\{1,2,\cdots,m\})\cup((0,1/n)\times\{m+1,m+2,\cdots\}))$$

It is therefore not ORC, but it is Alster since it is a quotient of an Alster space.

Again, it is easy to see that X is ORC-scattered. Indeed since every point but p has a compact neighbourhood, $D_{\mathrm{ORC}}(X)$ is a single point and $D^2_{\mathrm{ORC}}(X)=\emptyset$.

6.4 Example For the next example, we assume CH and give an example of a space that is Lindelöf and not productively Lindelöf, but has an uncountable discrete subspace whose complement is countable. This example contradicts [Abu Osma and Henriksen, 2004, Theorem 3.8] in which the eleventh line of the claimed proof interchanges a join and a meet without explanation.

It is also an example of an Alster-scattered space that is not Alster and shows that one cannot replace the ORC-scattered by Alster-scattered in Theorem 25.

Let $\mathbf{R}$ denote the space of reals with the usual topology and $\mathbf{R}_\nu$ the same pointset with a new topology that we describe below. We denote by $\mathbf{Q}$ the space of rationals with the usual topology.

In $\mathbf{R}_\nu$, every irrational point is open. A basic neighbourhood of a rational point q has the form $(a,b)-D$ where $a<q<b$ and D is a countable subset of irrational numbers. Since such a set is determined by the endpoints and a choice of D, the cardinality of such

basic opens is ω_1. It is clear that since D consists of irrational numbers, $\mathbf{Q}$ appears as a subspace of $\mathbf{R}_\nu$ with its usual topology.

Proposition 36. *$\mathbf{R}_\nu$ is regular.*

Proof. We will show that whenever U is open and $p \in U$, then there is an open set V such that $p \in V \subseteq \mathrm{cl}(V) \subseteq U$. Since each irrational is clopen this is clear when $p \notin \mathbf{Q}$. Now suppose $p \in \mathbf{Q}$. It is sufficient to consider the case that U is basic, so suppose $U = (a, b) - D$ as above. If c and d are chosen so that $a < c < p < d < b$ and c and d are irrational, then (c, d) and $(c, d) - D$ are closed since each irrational is clopen. Then $p \in (c, d) - D \subseteq (a, b) - D$ is the required sequence. □

The following is an obvious consequence of the Baire category theorem and is probably well known. We include it for completeness.

Proposition 37. *Every dense G_δ in $\mathbf{R}$ is uncountable.*

Proof. A dense G_δ is a countable meet of dense open sets. If it were countable, we could further intersect the G_δ set with the complements of the points of that countable set and then we would have an empty countable meet of dense open sets, which contradicts the Baire category theorem. □

Let us say that a countable set $\mathcal{B}$ of basic open sets of $\mathbf{R}_\nu$ (as defined above) is a **countable basic open cover of $\mathbf{Q}$** if it is a countable cover of $\mathbf{Q}$ in $\mathbf{R}_\nu$. Since there are ω_1-many basic open sets and such a cover is determined by a sequence of basic open sets, it is clear by CH that there are ω_1-many such countable basic open covers. Let us enumerate them as $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_\alpha, \ldots$, $\alpha < \omega_1$.

Proposition 38. *If $\mathcal{B}$ is a countable basic open cover of $\mathbf{R}_\nu$, then $\bigcup \mathcal{B}$ is a dense G_δ in $\mathbf{R}$.*

Proof. Suppose $\mathcal{B} = \{(a_n, b_n) - D_n) \mid n \in \mathbf{N}\}$ is a countable basic open cover. Then

$$\bigcup \mathcal{B} = \bigcup ((a_n, b_n) - D_n) = \bigcup (a_n, b_n) \cap \bigcap \{(\mathbf{R}_\nu - \{x\}) \mid x \in \bigcap D_n\}$$

which is the meet of an open set and a G_δ and hence a G_δ in $\mathbf{R}$. It is dense in $\mathbf{R}$ because it contains $\mathbf{Q}$. □

We will now choose inductively an ω_1-indexed sequence $t_1, t_2, \ldots, t_\alpha, \ldots$ of irrational numbers. We let t_1 be any irrational. Suppose we have chosen t_β for all $\beta < \alpha$. Since $\bigcup \mathcal{B}_\beta$ is a dense G_δ of $\mathbf{R}$ for all $\beta < \alpha$ and there are only countably many $\beta < \alpha$, it follows that $\bigcap_{\beta<\alpha} \bigcup \mathcal{B}_\beta$ is a dense G_δ and therefore uncountable. The set $\{t_\beta \mid \beta < \alpha\}$ is countable and hence we can choose some

$$t_\alpha \in \left(\bigcap_{\beta<\alpha} \bigcup \mathcal{B}_\beta \right) - \{t_\beta \mid \beta < \alpha\} - \mathbf{Q}$$

Now we let $X = \mathbf{Q} \cup \{t_\alpha \mid \alpha < \omega_1\}$ with the topology inherited from $\mathbf{R}_\nu$.

Proposition 39. *X is Lindelöf and completely regular.*

Proof. Any open cover has a refinement by basic opens. Let $\mathcal{O}$ be such a cover of X. Since $\mathcal{O}$ contains a cover of $\mathbf{Q}$, some $\mathcal{B}_\alpha \subseteq \mathcal{O}$. But by construction, $t_\gamma \in \bigcup \mathcal{B}_\alpha$ for every $\gamma > \alpha$. Thus $\mathcal{B}_\alpha$, together with sets in $\mathcal{O}$ that cover the countably many t_β for $\beta < \alpha$ is a countable refinement of $\mathcal{O}$. So X is Lindelöf, regular by Proposition 36, thus normal by [Kelley (1955), Lemma 4.1], and hence completely regular. □

Proposition 40. *X is not Alster and therefore, in the presence of CH, not productively Lindelöf.*

Proof. We begin by observing that, since each irrational is open, a compact set can contain only finitely many of them. A compact set in $\mathbf{Q}$ must be compact and therefore closed and a G_δ in the usual topology, which makes it a G_δ in R_ν. Thus the cover consisting of all the compact sets of $\mathbf{Q}$ and all the singletons of $X - \mathbf{Q}$ is an ample G_δ cover without a countable refinement. Since X has weight ω_1, it follows from [Alster (1988), 1.1], which assumes CH, that X cannot be productively Lindelöf. □

7 Some open questions.

1. Is productively Lindelof weaker than Alster?
2. If a space is S-scattered, must each D_S^α be nowhere dense in $D_S^{\alpha+1}$? (This is known to be true in the cases D and P.)
3. Is there an example of 6.4 that does not use CH?
4. In an earlier version of the paper, we had a condition called the compact P-support (CPS) which meant that for every point p, there is a compact set A with the property that every G_δ containing A contains a neighbourhood of p. This condition was intended to be a single condition that included both P-spaces and LC-spaces, was preserved by products, closed subspaces, etc. We then discovered the ORC that had all those properties and was implied by CPS. It is an open question whether CPS is strictly stronger than ORC.

References

[Abu Osma and Henriksen, 2004] E. Abu Osma and M. Henriksen (2004), *Essential P-spaces: a generalization of door spaces.* Comment. Math. Univ. Carolinae, **45**, 509–518.

[Alster (1988)] K. Alster (1988), *On the class of all spaces of weight not greater than ω_1 whose Cartesian product with every Lindelöf space is Lindelöf.* Fund. Math. **129**, 133–140.

[Barr, Kennison, & Raphael (2006)] M. Barr, J.F. Kennison, and R. Raphael (2006), *Searching for absolute $\mathcal{CR}$-epic spaces.* Accepted for Canadian J. Math.

[Engelking, (1989)] R. Engelking (1989), General Topology, Heldermann, Berlin.

[Gillman & Jerison (1960)] L. Gillman and M. Jerison (1960), *Rings of Continuous Functions.* D. Van Nostrand, Princeton.

[Henriksen *et. al.*, (to appear)] M. Henriksen, R. Raphael, and R. G. Woods, SP-scattered spaces. To appear.

[Kelley (1955)] J.L. Kelley (1955), *General Topology.* Van Nostrand, New York.

[Levy & Rice (1981)] R. Levy and M.D. Rice (1981), *Normal P-spaces and the G_δ topology.* Colloq. Math. **44**, 227–240.

[Telgarsky, 1971] R. Telgarsky (1971), C-scattered and paracompact spaces. Fund. Math, **73**, 59–74.