DESCRIPTIVE SET THEORY PROBLEM SET

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1. Let X be a second-countable topological space.

(a) Show that X has at most continuum-many open subsets.

(b) Let α, β, γ denote ordinals. A sequence of sets $(A_\alpha)_{\alpha<\gamma}$ is called *monotone* if it is either increasing (i.e. $\alpha < \beta \Rightarrow A_\alpha \subseteq A_\beta$, for all $\alpha, \beta < \gamma$) or decreasing (i.e. $\alpha < \beta \Rightarrow A_\alpha \supseteq A_\beta$, for all $\alpha, \beta < \gamma$); call it *strictly monotone*, if all of the inclusions are strict.

Prove that any strictly monotone sequence $(U_\alpha)_{\alpha<\gamma}$ of open subsets of X has countable length, i.e. γ is countable.

Hint: Use the same idea as in the proof of (a).

(c) Show that every monotone sequence $(U_\alpha)_{\alpha<\omega_1}$ open subsets of X eventually stabilizes, i.e. there is $\gamma < \omega_1$ such that for all $\alpha < \omega_1$ with $\alpha \geqslant \gamma$, we have $U_\alpha = U_\gamma$.

Hint: Use the regularity of ω_1, i.e. supremum of countably-many countable ordinals is still a countable ordinal.

(d) Conclude that parts (a), (b) and (c) are also true for closed sets.

2. Prove that any separable metric space has cardinality at most continuum.

Remark: This is true more generally for first-countable separable Hausdorff topological spaces, but false for general separable Hausdorff topological spaces (try to construct a counter-example).

3. Let X be a topological space. Prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

(1) X is a totally bounded metric space.

(2) X is a separable metric space.

(3) X is a second countable.

(4) X is Lindelöf, i.e. every open cover admits a countable subcover.

4. (a) Show that a metric space X is complete if and only if every decreasing sequence of closed sets $(B_n)_{n\in\mathbb{N}}$ with $\operatorname{diam}(B_n) \to 0$ has nonempty intersection (in fact, $\bigcap_{n\in\mathbb{N}} B_n$ is a singleton).

(b) Show that the requirement in (a) that $\operatorname{diam}(B_n) \to 0$ cannot be dropped. Do this by constructing a complete metric space that has a decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of closed **balls** with $\bigcap_{n\in\mathbb{N}} B_n = \emptyset$.

Date: October 3, 2021.

Many of the problems here are borrowed from "Classical Descriptive Set Theory" by A. Kechris.

HINT: Use $\mathbb{N}$ as the underlying set for your metric space.

5. Prove that a fiber product over a Hausdorff space is a closed subset of the product. Conclude that a fiber product of countably-many Polish spaces over a Hausdorff space is Polish.

6. By definition, the class of G_δ sets is closed under countable intersections. Show that it is also closed under finite unions. Equivalently, the class of F_σ sets is closed under finite intersections.

HINT: Think in terms of quantifiers $\forall$ and $\exists$ rather than intersections and unions; for example, if $A = \bigcap_n U_n$, then $x \in A \iff \forall n (x \in A_n)$.

7. [1] Prove that for every Polish space X, there is an injection $c : X \hookrightarrow \mathcal{C}$ such that the c-preimages of the sets $V_n := \{x \in \mathcal{C} : x(n) = 1\}$, $n \in \mathbb{N}$, are open. In particular, the c-preimages of open sets are F_σ.

HINT: Encode the points of X as binary sequences using a countable basis $(U_n)_{n\in\mathbb{N}}$.

8. [1] Prove that every Polish space X admits a linear ordering $<$ that is both G_δ and F_σ[2] as a subset of X^2.

HINT: Think of the points of X as binary sequences using a countable basis $(U_n)_{n\in\mathbb{N}}$.

9.
(a) Show that the Cantor set (with relative topology of $\mathbb{R}$) is homeomorphic to the Cantor space.

(b) Show that the Baire space $\mathcal{N}$ is homeomorphic to a G_δ subset of the Cantor space $\mathcal{C}$.

(c) Show that the set of irrationals (with the relative topology of $\mathbb{R}$) is homeomorphic to the Baire space.

HINT: Use the continued fraction expansion.

10. Let $T \subseteq A^{<\mathbb{N}}$ be a tree and suppose it is finitely branching. Prove that $[T]$ is compact.

11. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree. Define a total ordering $<$ on T such that $<$ is a well-ordering if and only if T doesn't have an infinite branch.

12. Let S, T be trees on sets A, B, respectively. Prove that for any G_δ set $D \subseteq [S]$ and continuous function $f : D \to [T]$ there is a monotone map $\varphi : S \to T$ such that $f = \varphi^*$. In particular, $\mathrm{dom}(f) = \mathrm{dom}(\varphi^*)$.

HINT: First solve for $D := [S]$: for $s \in S$, define $\varphi(s)$ to be the longest $t \in T$ such that $|t| \leqslant |s|$ and $N_t \supseteq f(N_s)$. For the general case, write $D = \bigcap\!\!\!\!\downarrow_n U_n$, where the U_n are open, and replace $|s|$ with the largest $n \leqslant |s|$ such that $N_s \cap D \subseteq U_n$. The case $N_s \cap D = \emptyset$ needs a special (yet straightforward) care.

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[1] Thanks to Jenna Zomback for this.

[2] Thanks to Wei Dai for pointing out that this is also F_σ.

13. Using the outline below, prove the following:

Proposition. *Let (X, d) be a metric space. The following are equivalent:*

(1) *X is compact.*

(2) *Every sequence in X has a convergent subsequence.*

(3) *X is complete and totally bounded.*

(4) *X is separable and every decreasing sequences of nonempty closed sets has an intersection.*

In particular, compact metrizable spaces are Polish.

(1) $\Rightarrow$ (2): For a sequence $(x_n)_n$, let K_m be the closure of the tail $\{x_n\}_{n \geqslant m}$ of the sequence and use the intersection-of-closed sets version of the definition of compactness.

(2) $\Rightarrow$ (3): For total boundedness, fix an $\varepsilon > 0$ and start constructing an ε-net F by adding elements to your F that are not yet covered by $B(F, \varepsilon)$. For completeness, note that if a subsequence of a Cauchy sequence converges, then so does the entire sequence.

(3) $\Rightarrow$ (4): Separability follows from total boundedness, see Question 3. Let (K_n) be a decreasing sequences of nonempty closed sets and $\varepsilon_n \to 0$. There is a finite collection $\mathcal{B}_0$ of balls or radius ε_0 that covers K_0. One of these balls has to intersect infinitely-many K_n.

(4) $\Rightarrow$ (1): Separability implies Lindelöf (i.e. every open cover has a countable subcover), see Question 3. Every countable open cover having a finite subcover is equivalent to every countable collection of closed sets with the finite intersection property having a nonempty intersection.

14. Let X be a compact metric space and Y be a separable complete metric space. Let $C(X, Y)$ be the space of continuous functions from X to Y equipped with the uniform metric, i.e. for $f, g \in C(X, Y)$,

$$d_u(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Prove that $C(X, Y)$ is a separable complete metric space, hence Polish.

HINT 1: Proving separability is tricky, so you may want to first prove it for $X = [0, 1]$ and $Y = \mathbb{R}$. In the general case (to prove separability), note that by uniform continuity,

$$C(X, Y) = \bigcup_n A_{n,m},$$

for every $n \in \mathbb{N}$, where

$$A_{n,m} = \{f \in C(X, Y) : \forall x, y \in X (d_X(x, y) < 1/n \Rightarrow d_Y(f(x), f(y)) < 1/m)\}.$$

Realize that it is enough to show that for any $n, m \in \mathbb{N}$, there is a countable $B_{n,m} \subseteq A_{n,m}$ such that for any $f \in A_{n,m}$ there is $g \in B_{n,m}$ with $d_u(f, g) < 3/m$. Now fix n, m and try to construct $B_{n,m}$; when doing so, don't try to *define* each function in $B_{n,m}$ by hand as you would maybe do in the case $X = [0, 1]$; instead, carefully *pick* them out of functions in $A_{n,m}$.

HINT 2: This is Theorem 4.19 in Kechris's "Classical Descriptive Set Theory".

15. Show that Hausdorff metric on $\mathcal{K}(X)$ is compatible with the Vietoris topology.

16. Let (X, d) be a metric with $d \leqslant 1$. For $(K_n)_n \subseteq \mathcal{K}(X) \setminus \{\emptyset\}$ and nonempty $K \in \mathcal{K}(X)$:

(a) $\delta(K, K_n) \to 0 \Rightarrow K \subseteq \underline{\mathrm{T}\lim}_n K_n$;

(b) $\delta(K_n, K) \to 0 \Rightarrow K \supseteq \overline{\mathrm{T}\lim}_n K_n$.

In particular, $d_H(K_n, K) \to 0 \Rightarrow K = \mathrm{T}\lim_n K_n$. Show that the converse may fail.

17. Let (X, d) be a compact metric with $d \leqslant 1$. For sequence $(K_n)_n \subseteq \mathcal{K}(X) \setminus \{\emptyset\}$, show the following:

(a) $\delta(\underline{\mathrm{T}\lim}_n K_n, K_m) \to 0$ as $m \to \infty$;

(b) $\delta(K_m, \overline{\mathrm{T}\lim}_n K_n) \to 0$ as $m \to \infty$.

Thus, if $K = \mathrm{T}\lim_n K_n$ exists, then $d_H(K_n, K) \to 0$.

18. Let (X, d) be a metric space with $d \leqslant 1$. Then $x \mapsto \{x\}$ is an isometric embedding of X into $\mathcal{K}(X)$.

19. Let (X, d) be a metric space with $d \leqslant 1$ and assume $K_n \to K$. Then any sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in K_n$ has a subsequence converging to a point in K.

20. Let X be metrizable.

(a) The relation "$x \in K$" is closed, i.e. $\{(x, K) : x \in K\}$ is closed in $X \times \mathcal{K}(X)$.

(b) The relation "$K \subseteq L$" is closed, i.e. $\{(K, L) : K \subseteq L\}$ is closed in $\mathcal{K}(X)^2$.

(c) The relation "$K \cap L \neq \emptyset$" is closed, i.e. $\{(K, L) : K \cap L \neq \emptyset\}$ is closed in $\mathcal{K}(X)^2$.

(d) The map $(K, L) \mapsto K \cup L$ from $\mathcal{K}(X)^2$ to $\mathcal{K}(X)$ is continuous.

(e) If Y is metrizable, then the map $(K, L) \mapsto K \times L$ from $\mathcal{K}(X) \times \mathcal{K}(Y)$ into $\mathcal{K}(X \times Y)$ is continuous.

(f) Find a compact X for which the map $(K, L) \mapsto K \cap L$ from $\mathcal{K}(X)^2$ to $\mathcal{K}(X)$ is not continuous.

21. Let X be a topological space.

(a) If X is nonempty perfect, then so is $\mathcal{K}(X) \setminus \{\emptyset\}$.

(b) If X is compact metrizable, then $C(X)$ is perfect, where $C(X) = C(X, \mathbb{R})$.

22. (AC) Show that any nonempty perfect compact Hausdorff space X has cardinality at least continuum by constructing an injection from the Cantor space into X.

Hint: Mimic the proof for Polish spaces.

23. Let X be a nonempty perfect Polish space and let Q be a countable dense subset of X. Show that Q is F_σ but not G_δ. In particular, $\mathbb{Q}$ is not Polish (in the relative topology of $\mathbb{R}$).

24. [3] Show that $[0,1]$ does not admit a countable nontrivial[4] partition into closed intervals.

HINT: What kind of subset would the endpoints of those intervals form?

25. Show that the perfect kernel of a Polish space X is the largest perfect subset of X, i.e. it contains all other perfect subsets.

26. A *topological group* is a group with a topology on it so that group multiplication $(x,y) \to xy$ and inverse $x \to x^{-1}$ are continuous functions. Show that a countable topological group is Polish if and only if it is discrete.

27. Let X be separable metrizable. Show that

$$\mathcal{K}_p(X) := \{K \in \mathcal{K}(X) : K \text{ is perfect}\}$$

is a G_δ set in $\mathcal{K}(X)$. In particular, if X is Polish, then so is $\mathcal{K}_p(X)$.

28. (a) Let X be a Polish space. Show that if $K \subseteq X$ is countable and compact, then its Cantor–Bendixson rank $|K|_{\boldsymbol{C}}$ is not a limit ordinal.

(b) For each nonlimit ordinal $\alpha < \omega_1$, construct a countable compact subset K_α of $\mathcal{C}$, whose Cantor–Bendixson rank is exactly α.

29. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree.

(a) Suppose T is pruned. Find a condition (on the nodes of the tree T) such that T satisfies it if and only if $[T]$ is perfect (as a subset of $\mathcal{N}$).

(b) Define a Cantor–Bendixson derivative T' of T, as well as the iterated derivatives $(T^\alpha)_{\alpha \in \mathrm{ON}}$, such that $[T^\infty]$ is the perfect kernel of $[T]$, i.e. $[T^\infty] = [T]^\infty$.

REMARK: The statement of this question is somewhat vague and informal, but understanding it is part of the challenge.

30. Let X be a second countable zero-dimensional space.

(a) Prove Kuratowski's reduction property: If $A, B \subseteq X$ are open, there are open $A^* \subseteq A, B^* \subseteq B$ with $A^* \cup B^* = A \cup B$ and $A^* \cap B^* = \emptyset$.

HINT: Write A and B as countable unions of clopen sets: $A = \bigcup_n A_n$, $B = \bigcup_n B_n$. Put those points x of A in A^* that are covered by A_n no later than by B_n, i.e. if n is the smallest number such that $x \in A_n \cup B_n$, then $x \in A_n$.

(b) Conclude the following separation property: For any disjoint closed sets $A, B \subseteq X$, there is a clopen set C separating A and B, i.e. $A \subseteq C$ and $B \cap C = \emptyset$.

31. (a) Let X be a nonempty zero-dimensional Polish space such that all of its compact subsets have empty interior. Fix a complete compatible metric and prove that there is a Luzin scheme $(A_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ with vanishing diameter and satisfying the following properties:

[3] Thanks to Jenna Zomback for sparking this problem.

[4] A partition $\mathcal{P}$ of a set X is *trivial* if $\mathcal{P} = \{X\}$.

(i) $A_\emptyset = X$;

(ii) A_s is nonempty clopen;

(iii) $A_s = \bigcup_{i \in \mathbb{N}} A_{s^\frown i}$.

HINT: Assuming A_s is defined, cover it by countably many clopen sets of diameter at most $\delta < 1/n$, and choose the δ small enough so that any such cover is necessarily infinite.

(b) Derive the Alexandrov–Urysohn theorem, i.e. show that the Baire space is the only topological space, up to homeomorphism, that satisfies the hypothesis of (a).

32. For this exercise, you may use the Alexandrov–Urysohn theorem without proof.

(a) Let $Y \subseteq \mathbb{R}$ be G_δ and such that Y, $\mathbb{R} \setminus Y$ are dense in $\mathbb{R}$. Show that Y is homeomorphic to $\mathcal{N}$.

(b) Show that part (a) may fail if $\mathbb{R}$ is replaced by $\mathbb{R}^2$.

(c) However, prove that part (a) holds if $\mathbb{R}$ is replaced by any zero-dimensional nonempty Polish space.

33. Show that for any Polish space X there is a continuous open surjection $g : \mathcal{N} \twoheadrightarrow X$ by constructing a sequence $(U_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ of open subsets of X such that

(i) $U_\emptyset = X$

(ii) $\overline{U}_{s^\frown i} \subseteq U_s$

(iii) $U_s = \bigcup_i U_{s^\frown i}$

(iv) $\operatorname{diam}(U_s) < 2^{-|s|}$.

CAUTION: We don't require $U_{s^\frown i} \cap U_{s^\frown j} = \emptyset$ for $i \neq j$ (which makes your life easy), so the associated map g may not be injective.

34. Using part (c) of Question 32, prove the following:[5]

Theorem (Strengthening of the Perfect Set Theorem)**.** *Every nonempty perfect Polish space contains a dense G_δ subset homeomorphic to the Baire space.*

35. The following steps outline a proof of the Baire category theorem for locally compact Hausdorff spaces.

1) Show that compact Hausdorff spaces are normal.

2) Using part (1), prove that in locally compact[6] Hausdorff space X, for every nonempty open set U and every point $x \in U$, there is a nonempty precompact[7] open $V \ni x$ with $\overline{V} \subseteq U$.

[5]Thanks to Anton Bernshteyn for suggesting this problem.

[6]A topological space is said to be *locally compact* if every point has a neighborhood basis that consists of precompact[7] open sets.

[7]*Precompact* sets are those contained in compact sets. For Hausdorff spaces, this is equivalent to having a compact closure.

3) Prove that locally compact Hausdorff spaces are Baire.

36. For topological space X, Y, a continuous map $f : X \to Y$ is called *category preserving* if f-preimages of meager sets are meager.

(a) Show that any continuous open map $f : X \to Y$ is category preserving (in fact, f-preimages of nowhere dense are nowhere dense). In particular, projections are category preserving.

(b) For topological spaces X, Y, if X is Baire, then, for a continuous map $f : X \to Y$, the following are equivalent:

(1) f is category preserving.

(2) f-preimages of nowhere dense sets are nowhere dense.

(3) f-images of open sets are somewhere dense.[8]

(4) f-preimages of dense open sets are dense.

37. [9]

(a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every irrational but discontinuous at every rational.

(b) Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every rational but discontinuous at every irrational.

HINT: Show that the set of continuity points of any function is G_δ.

38. Recall that $C([0, 1])$ is a Polish space with the uniform metric. Show that the generic element of $C([0, 1])$ is nowhere differentiable following the outline below.

1) Prove that given $m \in \mathbb{N}$, any function $f \in C([0, 1])$ can be approximated (in the uniform metric) by a piecewise linear function $g \in C([0, 1])$, whose linear pieces (finitely many) have slope $\pm M$, for some $M \geqslant m$.

2) For each $n \geqslant 1$, let E_n be the set of all functions $f \in C([0, 1])$, for which there is $x_0 \in [0, 1]$ (depending on f) such that $|f(x) - f(x_0)| \leqslant n|x - x_0|$ for all $x \in [0, 1]$. Show that E_n is nowhere dense using the fact that if g is as in (1) with $m = 2n$, then some open neighborhood of g is disjoint from E_n.

39. Let X be a perfect Polish space and show that a generic compact subset of X is perfect, i.e. show that the set $\mathcal{K}_p(X)$ is comeager in $\mathcal{K}(X)$ (see Question 27).

40. A *finite bounded game* on a set A is a game similar to infinite games, but the players play at most n number of steps before the winner is decided, for some fixed number $n \geqslant 1$ (say a million). More formally, the game is a tree $T \subseteq A^{<n}$, for some n, and the runs of the game are exactly the elements of the set $\mathrm{Leaves}(T)$ of all leaves of T, so the payoff set is a subset $D \subseteq \mathrm{Leaves}(T)$. Player I wins the run $s \in \mathrm{Leaves}(T)$ of the game

[8] Thanks to Dakota Ihli for suggesting this.

[9] Thanks to Francesco Cellarosi for bringing up the statements of this question to me.

iff $s \in D$. Consequently, Player II wins iff $s \in \text{Leaves}(T) \setminus D$. All games that appear in real life are such games, e.g. chess (counting ties as a win for Player II).

Prove the determinacy of finite bounded games.

Hint: Let's write down what it means for Player I to have a winning strategy in this game, assuming for simplicity that n is even and that all of the runs of the game are of length exactly n:

$$\exists a_1 \forall a_2 ... \exists a_{n-1} \forall a_n ((a_1, ..., a_n) \in D).$$

What happens when you negate this statement?

41. A *finite game* on a set A is a game similar to infinite games, but the players play only finitely many steps before the winner is decided. More formally, it is a (possibly infinite) tree $T \subseteq A^{<\mathbb{N}}$ that has no infinite branches, and the set of runs is $\text{Leaves}(T)$, so the payoff set is a subset $D \subseteq \text{Leaves}(T)$. Player I wins the run $s \in \text{Leaves}(T)$ of the game iff $s \in D$. Consequently, Player II wins iff $s \in \text{Leaves}(T) \setminus D$.

(a) Prove the determinacy of finite games.

Hint: Call a position $s \in T$ determined, if from that point on, one of the players has a winning strategy. Thus, no player has a winning strategy in the beginning iff $\emptyset$ is undetermined. What can you say about extensions of undetermined positions?

(b) Conclude the determinacy of clopen infinite games. (These are games with runs in $A^{\mathbb{N}}$ and the payoff set a clopen subset of $A^{\mathbb{N}}$.)

42. [10] Let $\mathcal{G}$ be the so-called *Hamming graph* on $\mathcal{C}$, namely, there is an edge between $x, y \in \mathcal{C}$ exactly when x and y differ by one bit.

(a) Prove that $\mathcal{G}$ is has no odd cycles and hence is bipartite (admits a 2-coloring). Pinpoint the use of AC.

(b) Fix a coloring $c : \mathcal{C} \to 2$ of $\mathcal{G}$ and let $A_i := c^{-1}(i)$ for $i \in \{0, 1\}$. Consider the game where each player plays a finite nonempty binary sequence at each step and a play is the concatenation of those finite sequences, thus an infinite binary sequence. Prove that this game with the payoff set A_0 is not determined by showing that if one of the players had a winning strategy, so would the other one.

Hint: Steal the other player's strategy.

43. In ZF (in particular, don't use AC or $\neg$AD), define a game with rules $G(T, D)$ on the set $A = \mathscr{P}(\mathbb{N}^{\mathbb{N}})$ (i.e. define a pruned tree $T \subseteq A^{<\mathbb{N}}$ and a set $D \subseteq A^{\mathbb{N}}$), so that ZF+$\neg$AD implies that this game is undetermined. In other words, you have to define the tree T and the payoff set D without using $\neg$AD, but then prove that the game $G(T, D)$ is undetermined using $\neg$AD.

Hint: Note that besides playing subsets of $\mathbb{N}^{\mathbb{N}}$, players can also play natural numbers in the sense that $\mathbb{N} \hookrightarrow \mathscr{P}(\mathbb{N}^{\mathbb{N}})$ by $n \mapsto \{(n)_{i \in \mathbb{N}}\}$.

[10] Thanks to Forte Shinko for suggesting this problem.

44. Let X be a second countable Baire space. Show that the σ-ideal $\mathrm{MGR}(X)$ has the countable chain condition in $\mathrm{BMEAS}(X)$, i.e. there is no uncountable family $\mathcal{A} \subseteq \mathrm{BMEAS}(X)$ of nonmeager sets such that for any two distinct $A, B \in \mathcal{A}$, $A \cap B$ is meager.

45. Let X be a topological space.

(a) If $A_n \subseteq X$, then for any open $U \subseteq X$,
$$U \Vdash \bigcap_n A_n \iff \forall n (U \Vdash A_n).$$

(b) If X is a Baire space, A is Baire measurable, and $U \subseteq X$ is nonempty open, then
$$U \Vdash A^c \iff \forall V \subseteq U (V \nVdash A),$$
where V varies over a weak basis[11] for X.

(c) If X is a Baire space, the sets $A_n \subseteq X$ are Baire measurable, and U is nonempty open, then
$$U \Vdash \bigcup_n A_n \iff \forall V \subseteq U \exists W \subseteq V \exists n (W \Vdash A_n).$$
where V, W vary over a weak basis for X.

46. Prove that a topological group G is Baire iff G is nonmeager.

47. Let X be a topological space and $A \subseteq X$.

(a) Show that $U(A)$ is regular open, i.e. it is equal to the interior of its closure.

(b) If moreover X is a Baire space and A is Baire measurable, then $U(A)$ is the unique regular open set U with $A =^* U$.

48. Let G be a Polish group (i.e. a topological group whose topology happens to be Polish) and let $H < G$ be a subgroup. Prove that H is Polish iff H is closed.

Hint: Consider H inside $\overline{H}$. What is the Baire category status (meager/nonmeager/comeager) of H in (the relative topology of) $\overline{H}$? If $H \subsetneq \overline{H}$, look at the cosets.

49. Let Γ be a group acting on a Polish space X by homeomorphisms (i.e. each element $\gamma \in \Gamma$ acts as a homeomorphism of X). A set $A \subseteq X$ is called invariant if $\gamma A = A$ for all $\gamma \in \Gamma$. The action $\Gamma \curvearrowright X$ is called *generically ergodic* if every invariant Baire measurable set $A \subseteq X$ is either meager or comeager. For a set $A \subseteq X$, denote by $[A]_\Gamma$ the saturation of A, namely $[A]_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma A$.

Prove that the following are equivalent:

(1) $\Gamma \curvearrowright X$ is generically ergodic.

(2) Every invariant nonempty open set is dense.

(3) For comeager-many $x \in X$, the orbit $[x]_\Gamma$ is dense.

[11] A *weak basis* for a topological space X is a collection $\mathcal{V}$ of nonempty open sets such that every nonempty open set $U \subseteq X$ contains at least one $V \in \mathcal{V}$.

(4) There is a dense orbit.
(5) For every nonempty open sets $U, V \subseteq X$, there is $\gamma \in \Gamma$ such that $(\gamma U) \cap V \neq \emptyset$.

HINT: For (2)⇒(3), take a countable basis $\{U_n\}_{n \in \mathbb{N}}$ and consider $\bigcap_n [U_n]_\Gamma$.

50. Show that the Kuratowski–Ulam theorem fails if A is not Baire measurable by constructing a nonmeager set $A \subseteq \mathbb{R}^2$ (using AC) so that no three points of A are on a straight line.

HINT: Note that there are only continuum many F_σ sets, so take a transfinite enumeration $(F_\xi)_{\xi < 2^{\aleph_0}}$ of all *meager* F_σ sets, and construct a sequence $(a_\xi)_{\xi < 2^{\aleph_0}}$ of points in $\mathbb{R}^2$ by transfinite recursion so that for each $\xi < 2^{\aleph_0}$,

$$\{a_\lambda : \lambda \leqslant \xi\} \nsubseteq F_\xi,$$

and no three of the points in $\{a_\lambda : \lambda \leqslant \xi\}$ lie on a straight line.

HINT: Recall that in perfect Polish spaces (such as $\mathbb{R}, \mathbb{R}^2$), any nonmeager Baire measurable subset contains a copy of the Cantor space (this is because it contains a nonmeager G_δ set). Now if $A := \{a_\lambda : \lambda \leqslant \xi\} \subseteq F_\xi$, apply Kuratowski–Ulam to F_ξ to find $x \in \mathbb{R}$ such that $(F_\xi)_x$ is meager and the vertical line $L_x = \{(x, y) \in \mathbb{R} : y \in \mathbb{R}\}$ is disjoint from A.

51. Show that if X, Y are second countable Baire spaces, so is $X \times Y$.

52. **Definition.** A *filter* on a set X is a set $\mathcal{U} \subseteq \mathscr{P}(X)$ such that

(i) (Nontriviality) $X \in \mathcal{U}$ but $\emptyset \notin \mathcal{U}$;
(ii) (Upward closure) $A \in \mathcal{U}, B \supseteq A \Rightarrow B \in \mathcal{U}$;
(iii) (Closure under finite intersections) $A, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$.

A filter $\mathcal{U}$ is called an *ultrafilter* if $A \notin \mathcal{U} \Rightarrow A^c \in \mathcal{U}$ for every $A \subseteq X$. Finally, an ultrafilter is called *principal* if for some $x \in X$, $\{x\} \in \mathcal{U}$ (or, equivalently, $\mathcal{U} = \{A \subseteq X : x \in A\}$).

It is useful to think of a filter $\mathcal{U}$ as the family of all conull sets of a $\{0, 1\}$-valued finitely additive measure $\mu_\mathcal{U}$ on a subalgebra of $\mathscr{P}(X)$. In other words, sets in $\mathcal{U}$ should be thought of as large sets. $\mathcal{U}$ being an ultrafilter simply means that $\mu_\mathcal{U}$ is defined on all of $\mathscr{P}(X)$; in other words, if a set is not large then it is small (i.e. there are no intermediate sets). Also, $\mathcal{U}$ being principal means that $\mu_\mathcal{U}$ is a Dirac measure (i.e. a point-mass at some point x).

(a) (AC) Prove that for every infinite set X, there exists a nonprincipal ultrafilter on X; do it by showing that every filter is contained in an ultrafilter and applying this to the filter of cofinite sets (called the Fréchet filter).
(b) Identifying $\mathscr{P}(\mathbb{N})$ with $\mathcal{C} = 2^\mathbb{N}$, view ultrafilters on $\mathbb{N}$ as subsets of $\mathcal{C}$ and show that no nonprincipal ultrafilter $\mathcal{U}$ is Baire measurable (as a subset of $\mathcal{C}$).

53. Using the outline below, prove Pettis's theorem:

Theorem (Pettis)**.** *Let G be a topological group and $A \subseteq G$ be Baire measurable. If A is nonmeager, then $A^{-1}A$ contains an open neighborhood of the identity 1_G; in fact if $U \Vdash A$, then $U^{-1}U \subseteq A^{-1}A$.*

1) By Question 46, G must be Baire.
2) Note that for any sets $B, C \subseteq G$,
$$B \subseteq C^{-1}C \iff \forall h \in B \ (Ch \cap C \neq \emptyset). \tag{$*$}$$
3) Let $U \subseteq G$ be nonempty open such that $U \Vdash A$. Fix arbitrary $g \in U$ and note that $V := g^{-1}U \subseteq U^{-1}U$ is an open neighborhood of 1_G. Thus, by $(*)$, $\forall h \in V$, $Uh \cap U \neq \emptyset$.
4) Conclude that for each $h \in V$, $Ah \cap A \neq \emptyset$, and hence, by $(*)$ again, $V \subseteq A^{-1}A$.
5) Note that we have shown $g^{-1}U \subseteq A^{-1}A$ for arbitrary $g \in U$, and thus, $U^{-1}U \subseteq A^{-1}A$.

54. Let G be a Baire topological group (i.e. G is nonmeager) and let $H < G$ be a Baire measurable subgroup. Prove if H is nonmeager then it is actually clopen! In particular, if H has countable index in G, then it is clopen.

55. (a) **Automatic continuity:** Let G, H be topological groups, where G is Baire and H is separable. Then every Baire measurable group homomorphism $\varphi : G \to H$ is actually continuous!

Hint: Enough to prove continuity at 1_G, so let $U \ni 1_H$ be open and take an open neighborhood $V \ni 1_H$ such that $V^{-1}V \subseteq U$. Using the separability of H, deduce that $\varphi^{-1}(hV)$ is nonmeager for some $h \in H$ and apply Pettis's theorem.

(b) Conclude that if $f : (\mathbb{R}, +) \to (\mathbb{R}, +)$ is a Baire measurable group homomorphism, then for some $a \in \mathbb{R}$, $f(x) = ax$ for all $x \in \mathbb{R}$.

Hint: First show this for integers, then for rationals, etc.

56. Letting $d_A : \mathbb{R} \to [0, 1]$ denote the Lebesgue density function for $A \subseteq \mathbb{R}$, define $D(A) := \{x \in \mathbb{R} : d_A(x) = 1\}$. Show that Lebesgue measurable subsets $A \subseteq \mathbb{R}$ with $A \subseteq D(A)$ form a topology.

Hint: To show that a possibly uncountable union $A := \bigcup_{\ell \in L} A_\ell$ of such sets is still Lebesgue measurable, prove that it can be approximated from above by open sets. First reduce it to the case $A \subseteq (0, 1)$. Then for each $\varepsilon > 0$, let $\mathcal{C}$ be the collection of all intervals $I \subseteq (0, 1)$ that admit an $\ell \in L$ with $\frac{\lambda(A_\ell \cap I)}{\lambda(I)} > 1 - \varepsilon$. Show that $\mathcal{C}$ covers A and, in fact, there is a subcover $\mathcal{C}' \subseteq \mathcal{C}$ of A with $\lambda\left(\bigcup \mathcal{C}' \setminus A\right) < 2\varepsilon$. For the latter, use the version of the Vitali covering lemma as in Lemma 8 of the author's note on Lebesgue differentiation.

57. Prove the following facts about the density topology on $\mathbb{R}$. (Below λ denotes the Lebesgue measure on $\mathbb{R}$ and all topological terms are with respect to the density topology.)

(a) Every nonempty open set has positive measure.

(b) For a Lebesgue measurable set $A \subseteq \mathbb{R}$, explicitly compute $\mathrm{Int}(A)$ and $\overline{A}$, and conclude that $\lambda(\mathrm{Int}(A)) = \lambda(A) = \lambda(\overline{A})$.

58. Consider $\mathbb{R}$ with the density topology and Lebesgue measure λ. For $A \subseteq \mathbb{R}$, prove that the following are equivalent:

(1) A is nowhere dense in the density topology;

(2) A is meager in the density topology;

(3) A is λ-null.

Conclude that A is Baire measurable in the density topology if and only if it is Lebesgue measurable.

59. Let $X = \mathbb{I}^{\mathbb{N}}$ and put $C_0 = \{(x_n)_{n \in \mathbb{N}} : x_n \to 0\}$. Show that C_0 is in $\mathbf{\Pi}^0_3(X)$.

60. Let X be a topological space, $Y \subseteq X$, and let ξ be an ordinal with $1 \leqslant \xi < \omega_1$. Prove the following:

(a) If $\mathbf{\Gamma}$ is one of $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi, \mathcal{B}$, then $\mathbf{\Gamma}(Y) = \mathbf{\Gamma}(X)|_Y := \{A \cap Y : A \in \mathbf{\Gamma}(X)\}$.

(b) We also always have $\mathbf{\Delta}^0_\xi(Y) \supseteq \mathbf{\Delta}^0_\xi(X)|_Y$. If moreover, $Y \in \mathbf{\Delta}^0_\xi(X)$, then we also have $\mathbf{\Delta}^0_\xi(Y) \subseteq \mathbf{\Delta}^0_\xi(X)|_Y$. However, give an example of a Polish space X and $Y \subseteq X$ such that the last inclusion is false for $\xi = 1$.

61. A class $\mathbf{\Gamma}$ of sets is called *self-dual* if it is closed under complements, i.e. $\neg\mathbf{\Gamma} = \mathbf{\Gamma}$. Show that if $\mathbf{\Gamma}$ is a self-dual class of sets in topological spaces that is closed under continuous preimages, then for any topological space X there does not exist an X-universal set for $\mathbf{\Gamma}(X)$. Conclude that neither the class $\mathcal{B}(X)$ of Borel sets, nor the classes $\mathbf{\Delta}^0_\xi(X)$, can have X-universal sets.

62. Letting X be a separable metrizable space and $\lambda < \omega_1$ be a limit ordinal, put

$$\mathbf{\Omega}^0_\lambda(X) := \bigcup_{\xi<\lambda} \mathbf{\Sigma}^0_\xi(X) \;\Big(= \bigcup_{\xi<\lambda} \mathbf{\Delta}^0_\xi(X) = \bigcup_{\xi<\lambda} \mathbf{\Pi}^0_\xi(X)\Big).$$

(a) Let Y be an uncountable Polish space and prove that there exists a set $P \in \mathbf{\Delta}^0_\lambda(Y \times X)$ that parameterizes $\mathbf{\Omega}^0_\lambda(X)$.

Hint: First construct such a set for $Y = \mathbb{N} \times \mathcal{C}$. Then conclude it for $Y = \mathcal{C}$ using the fact that the following functions are continuous: $()_0 : \mathcal{C} \to \mathbb{N}$ and $()_1 : \mathcal{C} \to \mathcal{C}$ defined for $y \in \mathcal{C}$ by

$$y = 1^{(y)_0} {}^\frown 0 {}^\frown (y)_1.$$

Finally, conclude the statement for any Y using the perfect set property.

(b) Conclude that if X is uncountable Polish, then $\mathbf{\Delta}^0_\lambda(X) \supsetneq \mathbf{\Omega}^0_\lambda(X)$.

63. (a) Show that any Polish space admits a finer Polish topology that is zero-dimensional and has the same Borel sets, i.e. for a given Polish space $(X, \mathcal{T})$, there exists a zero-dimensional Polish topology $\mathcal{T}_0 \supseteq \mathcal{T}$ such that $\mathcal{B}(\mathcal{T}_0) = \mathcal{B}(\mathcal{T})$.

(b) Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be Polish and $f : X \to Y$ a Borel isomorphism. Show that there are Polish topologies $\mathcal{T}'_X \supseteq \mathcal{T}_X, \mathcal{T}'_Y \supseteq \mathcal{T}_Y$ with $\mathcal{B}(\mathcal{T}'_X) = \mathcal{B}(\mathcal{T}_X)$, $\mathcal{B}(\mathcal{T}'_Y) = \mathcal{B}(\mathcal{T}_Y)$ such that $f : (X, \mathcal{T}'_X) \to (Y, \mathcal{T}'_Y)$ is a homeomorphism. Moreover, $\mathcal{T}'_X, \mathcal{T}'_Y$ can be taken to be zero-dimensional.

(c) Let Γ be a countable group and consider a Borel action of Γ on a Polish space $(X, \mathcal{T})$, i.e. each $g \in \Gamma$ acts as a Borel automorphism of X. Prove that there exists a Polish topology $\mathcal{T}_0 \supseteq \mathcal{T}$ with $\mathcal{B}(\mathcal{T}_0) = \mathcal{B}(\mathcal{T})$ that makes the action of Γ continuous. Moreover, $\mathcal{T}_0$ can be taken to be zero-dimensional.

64. Let X, Y be topological spaces. If $f : X \to Y$ is continuous with respect to some coarser (not necessarily strictly) Hausdorff topology on Y, then $\mathrm{graph}(f)$ is closed in $X \times Y$.

65. Let X be a Tychonoff topological space. Prove that the topology on X generated by countably-many Polish topologies refining the original topology of X is Polish.

66. Let G be a group equipped with a Polish topology that makes multiplication continuous. Prove that G is a topological group, i.e., the inverse is continuous as well.

HINT: The inverse map is a group isomorphism from G to the opposite group[12] and its graph is nice.

67. Let X, Y be topological spaces and let $\mathrm{proj}_X : X \times Y \to X$ be the projection function. Prove the following statements:

(a) proj_X is continuous and open.

(b) proj_X does not in general map closed sets to closed sets, even for $X = Y = \mathbb{R}$.
REMARK: We will see shortly in the course that for certain $Y = \mathcal{N}$, the projection of a closed set may not even be Borel in general.

(c) For $X = Y = \mathbb{R}$, proj_X maps closed sets to σ-compact (and hence F_σ) sets. More generally, images of F_σ sets under continuous functions from σ-compact to Hausdorff spaces are F_σ.

(d) **Tube lemma:** If Y is compact, then proj_X indeed maps closed sets to closed sets.

HINT: It is perhaps tempting to use sequences, but this would only work for first-countable spaces. Instead, use the open cover definition of compact and show that for closed $F \subseteq X \times Y$, every point $x \in X \setminus \mathrm{proj}_X(F)$ has an open neighborhood disjoint from $\mathrm{proj}_X(F)$. The "correct" solution should use nothing but definitions.

68. Show that the class of analytic sets is closed under

(a) continuous images,

(b) continuous preimages,

(c) countable unions,

(d) countable intersections.

[12] The *opposite group* G^{op} of G is the group with the same underlying set, but the order of multiplication is switched: $x \cdot_{\mathrm{op}} y := y \cdot x$.

Show that these statements also hold if we replace "continuous" by "Borel".

69. Let X be Polish and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of disjoint analytic sets in X. Prove that there are disjoint Borel sets $\{B_n\}_{n\in\mathbb{N}}$ with $B_n \supseteq A_n$.

70. Let X, Y be Polish and $f : X \to Y$ Borel. Show that for $A \subseteq f(X)$, if $f^{-1}(A)$ is Borel, then A is Borel relative to $f(X)$, i.e. there is a Borel $A' \subseteq Y$ such that $A = A' \cap f(X)$.

71. Let X be Polish and let E be an analytic equivalence relation on X, i.e. E is an analytic subset of X^2.

(a) Show that for an analytic set A, its saturation $[A]_E \coloneqq \{x \in X : \exists y \in A(x \, E \, y)\}$ is also analytic.

(b) Let $A, B \subseteq X$ be disjoint E-invariant analytic sets (i.e., $[A]_E = A$, $[B]_E = B$). Prove that there is an E-invariant Borel set D separating A and B, i.e., $D \supseteq A$ and $D \cap B = \emptyset$.

72. Construct an example of a closed equivalence relation E on a Polish space X and a closed set $C \subseteq X$ such that the saturation $[C]_E$ is analytic but not Borel.

Remark: This shows that in part (a) of the previous question, "analytic" is the best we can hope for.

Hint: Take analytic $A \subseteq \mathcal{N}$ that's not Borel and let $C \subseteq \mathcal{N}^2$ be a closed set projecting down onto A. Define an appropriate equivalence relation E on $\mathcal{N}^2$ (i.e. $E \subseteq \mathcal{N}^2 \times \mathcal{N}^2$).

73. Let X be set and let $\mathcal{T}, \mathcal{T}'$ be Polish topologies on X such that $\mathcal{T} \subseteq \mathcal{B}(\mathcal{T}')$ (for example, this would hold if $\mathcal{T} \subseteq \mathcal{T}'$). Show that $\mathcal{B}(\mathcal{T}) = \mathcal{B}(\mathcal{T}')$.

74. Let $(G, \mathcal{T})$ be a Polish group, where $\mathcal{T}$ denotes the topology of G. Prove that if a Baire measurable subgroup $H < G$ admits a different topology $\mathcal{T}' \neq \mathcal{T}|_H$ with $\mathcal{T} \subseteq \mathcal{B}(\mathcal{T}')$ or $\mathcal{T}' \subseteq \mathcal{B}(\mathcal{T})$ (for example, this would hold if $\mathcal{T}'$ is strictly finer/coarser than $\mathcal{T}$) such that $(H, \mathcal{T}')$ is a Polish group, then H is meager in G (in the topology $\mathcal{T}$).

Hint: Recall Questions 54 and 55(a).

75. Prove the following characterization of Borel sets: A subset B of a Polish space X is Borel iff it is an injective continuous image of a closed subset of $\mathcal{N}$.

76. Let X be Polish and consider the coding map $c : \mathcal{F}(X) \to 2^{\mathbb{N}}$ defined by $F \mapsto$ the characteristic function of $\{n \in \mathbb{N} : F \cap U_n \neq \emptyset\}$. Prove that for $x \in 2^{\mathbb{N}}$, $x \in c(\mathcal{F}(X))$ if and only if

$$\forall U_n \subseteq U_m [x(n) = 1 \to x(m) = 1]$$

and

$$\forall U_n \forall \varepsilon \in \mathbb{Q}^+ \big[x(n) = 1 \to \exists \overline{U_m} \subseteq U_n \text{ with } \operatorname{diam}(U_m) < \varepsilon \text{ such that } x(m) = 1)\big].$$

Conclude that $c(\mathcal{F}(X))$ is a G_δ subset of $2^{\mathbb{N}}$ and hence the Effros space $\mathcal{F}(X)$ is standard Borel.

77. Let X be a Polish space. Show that $\mathcal{K}(X)$ is a Borel subset of $\mathcal{F}(X)$.[13]

78. Let X be a Polish space. A function $s : \mathcal{F}(X) \to X$ is called a selector if $s(F) \in F$ for every nonempty $F \in \mathcal{F}(X)$. The goal of this question is to show that for every Polish space X, the Effros Borel space $\mathcal{F}(X)$ admits a Borel selector.

(a) Show that $\mathcal{F}(\mathcal{N})$ admits a Borel selector.

(b) By Question 33, there is a continuous open surjection $g : \mathcal{N} \to X$. Prove that the map $f : \mathcal{F}(X) \to \mathcal{F}(\mathcal{N})$ defined by $F \mapsto g^{-1}(F)$ is Borel.

(c) Conclude that $\mathcal{F}(X)$ admits a Borel selector.

79. Let X, Y be Polish spaces and let $f : X \to Y$ be a continuous function such that $f(X)$ is uncountable. Put

$$\mathcal{K}_f(X) = \{K \in \mathcal{K}(X) : f|_K \text{ is injective}\},$$

and note that, for a fixed countable basis $\mathcal{U}$ of X and for $K \in \mathcal{K}(X)$,

$$K \in \mathcal{K}_f(X) \iff \forall U_1, U_2 \in \mathcal{U} \text{ with } \overline{U_1} \cap \overline{U_2} = \emptyset [f(\overline{U_1} \cap K) \cap f(\overline{U_2} \cap K) = \emptyset].$$

Next, show that for fixed $U_1, U_2 \in \mathcal{U}$ with $\overline{U_1} \cap \overline{U_2} = \emptyset$ the set

$$\mathcal{V} = \left\{K \in \mathcal{K}(X) : f(\overline{U_1} \cap K) \cap f(\overline{U_2} \cap K) = \emptyset\right\}$$

is open in $\mathcal{K}(X)$, and hence $\mathcal{K}_f(X)$ is G_δ.

80. Let X be a Polish space, $F \subseteq X \times \mathcal{N}$ and $A = \mathrm{proj}_X(F)$. Show that if Player II has a winning strategy in the unfolded Banach–Mazur game $G^{**}(F, X)$, then A is meager.

81. For a topological space X, show that BMEAS(X) admits envelopes: for a given $A \subseteq X$, first find a BMEAS(X)-envelope for it in terms of $U(\cdot)$, then write down explicitly what the set is.

82. Let X be a Polish space and let $\mathbf{C}(X)$ denote the smallest σ-algebra on X containing $\mathcal{B}(X)$ and closed under the operation $\mathcal{A}$.

(a) Show that $\sigma(\mathbf{\Sigma}^1_1(X)) \subseteq \mathcal{A}\mathbf{\Pi}^1_1(X) \subseteq \mathbf{C}(X)$.

Hint: For $\sigma(\mathbf{\Sigma}^1_1(X)) \subseteq \mathcal{A}\mathbf{\Pi}^1_1(X)$, it is enough to show that $\mathcal{A}\mathbf{\Pi}^1_1(X)$ is closed under countable unions and countable intersections. For countable unions, use the natural bijection $\mathbb{N}^{<\mathbb{N}} \times \mathbb{N} \xrightarrow{\sim} \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ given by $(n, s) \mapsto n^\frown s$. For countable intersections, use the usual diagonal (snake-like) bijection $\mathbb{N}^2 \xrightarrow{\sim} \mathbb{N}$ to monotonically encode finite sequences of elements of $\mathbb{N}^{<\mathbb{N}}$ into single elements of $\mathbb{N}^{<\mathbb{N}}$.

(b) For each uncountable Polish space Y show that there is a Y-universal set for $\mathcal{A}\mathbf{\Pi}^1_1(X)$.

Hint: Enough to prove for $Y = \mathcal{N}^{\mathbb{N}^{<\mathbb{N}}}$ (why?). Start with a $\mathcal{N}$-universal set $F \subseteq \mathcal{N} \times X$ for $\mathbf{\Pi}^1_1(X)$ and for each $s \in \mathbb{N}^{<\mathbb{N}}$, consider the set $P_s \subseteq \mathcal{N}^{\mathbb{N}^{<\mathbb{N}}} \times X$ defined as follows: for $(y, x) \in \mathcal{N}^{\mathbb{N}^{<\mathbb{N}}} \times X$, put $(y, x) \in P_s :\Leftrightarrow (y(s), x) \in F$.

(c) Conclude that for uncountable X, $\sigma(\mathbf{\Sigma}^1_1(X)) \subsetneq \mathcal{A}\mathbf{\Pi}^1_1(X) \subsetneq \mathbf{C}(X)$.

[13] Thanks to Anton Bernshteyn for suggesting this question.

83. (Fun problem) Prove directly (without using Wadge's theorem or lemma) that any countable dense $Q \subseteq 2^{\mathbb{N}}$ is $\boldsymbol{\Sigma}^0_2$-complete, by showing that player II has a winning strategy in the Wadge game $G_W(A, Q)$ for any $A \in \boldsymbol{\Sigma}^0_2(\mathcal{N})$.

84. For a property $P \subseteq \mathbb{N}$ of natural numbers, we use the following abbreviations:

$$\begin{aligned} \forall^\infty n P(n) \;&:\Leftrightarrow\; \{n \in \mathbb{N} : P(n)\} \text{ is cofinite } \Leftrightarrow \text{ for large enough } n,\ P(n) \text{ holds} \\ \exists^\infty n P(n) \;&:\Leftrightarrow\; \{n \in \mathbb{N} : P(n)\} \text{ is infinite } \Leftrightarrow \text{ for arbitrarily large } n,\ P(n) \text{ holds} \end{aligned}$$

Show that the set $Q_2 = \big\{x \in 2^{\mathbb{N}} : \forall^\infty n(x(n) = 0)\big\}$ is $\boldsymbol{\Sigma}^0_2$-complete and conclude that the set $N_2 = \big\{x \in 2^{\mathbb{N}} : \exists^\infty n(x(n) = 0)\big\}$ is $\boldsymbol{\Pi}^0_2$-complete.

85. Show that the following sets are $\boldsymbol{\Pi}^0_3$-complete:

(a) $P_3 = \big\{x \in 2^{\mathbb{N}\times\mathbb{N}} : \forall n \forall^\infty m(x(n, m) = 0)\big\}$,

HINT: Use Q_2 from the previous question.

(b) $C_3 = \big\{x \in \mathbb{N}^{\mathbb{N}} : \lim_n x(n) = \infty\big\}$.

HINT: Reduce P_3 to C_3.

86. Each binary relation on $\mathbb{N}$ is an element of $\mathscr{P}(\mathbb{N}^2)$, which we may identify with $2^{\mathbb{N}^2}$. Thus, we can define

$$\begin{aligned} \mathrm{LO} &= \Big\{x \in 2^{\mathbb{N}^2} : x \text{ is a linear ordering}\Big\} \\ \mathrm{WO} &= \Big\{x \in 2^{\mathbb{N}^2} : x \text{ is a well-ordering}\Big\}. \end{aligned}$$

(a) Show that LO is a closed subset of $2^{\mathbb{N}^2}$ and that WO is co-analytic.

(b) Prove that WO is actually $\boldsymbol{\Pi}^1_1$-complete.

HINT: Define an appropriate ordering on a tree to show that $\mathrm{WF} \leqslant_W \mathrm{WO}$, where $\mathrm{WF} = \mathrm{Tr} \setminus \mathrm{IF}$.

87. Prove the Schröder–Bernstein theorem for equivalence relations E, F on Polish spaces X, Y, respectively; that is: if $E \sqsubseteq^i_B F$ and $F \sqsubseteq^i_B E$, then $E \simeq_B F$.

88. **Odometer.** Let $X_0 = \big\{x \in 2^{\mathbb{N}} : \forall^\infty n\ x(n) = 0\big\}$, $X_1 = \big\{x \in 2^{\mathbb{N}} : \forall^\infty n\ x(n) = 1\big\}$, and put $X = 2^{\mathbb{N}} \setminus (X_0 \cup X_1)$. Note that X_0 and X_1 are $\mathbb{E}_0$-classes, so all we did is throwing away from $2^{\mathbb{N}}$ two $\mathbb{E}_0$-classes. Define a continuous action of $\mathbb{Z}$ on X so that the induced orbit equivalence relation $E_{\mathbb{Z}}$ is exactly $\mathbb{E}_0|_X$.

89. **Universality of the shift action.** Let $\Gamma \curvearrowright X$ be a Borel action of a countable group Γ on a Polish space X. Show that there is a Borel equivariant[14] embedding $f : X \hookrightarrow (2^{\mathbb{N}})^\Gamma$, where $\Gamma \curvearrowright (2^{\mathbb{N}})^\Gamma$ by shift as follows: $\gamma \cdot y(\delta) = y(\delta\gamma)$, for $\gamma, \delta \in \Gamma$, $y \in (2^{\mathbb{N}})^\Gamma$. In particular, f is a Borel reduction of the induced orbit equivalence relations.

[14] A map is called *equivariant* if it commutes with the action, i.e.. $\gamma \cdot f(x) = f(\gamma \cdot x)$, for $x \in X$.

90. (a) For $\ell : \mathbb{N} \to \mathbb{N}$, let $\mathbb{E}_0(\ell)$ be the restriction of $\mathbb{E}_0(\mathbb{N})$ to $\mathcal{N}_{\leqslant \ell} := \{x \in \mathcal{N} : x(n) \leqslant \ell(n)\}$. Show that $\mathbb{E}_0(\ell) \sqsubseteq_c \mathbb{E}_0$

(b) More generally, prove that $\mathbb{E}_0(\mathbb{N}) \sqsubseteq_c \mathbb{E}_0$.

(c) Show that $\mathbb{E}_v \sim_B \mathbb{E}_0$ by proving that $\mathbb{E}_v \sqsubseteq_B \mathbb{E}_0(\mathrm{id}_{\mathbb{N}}) \sqsubseteq_c \mathbb{E}_0 \sqsubseteq_c \mathbb{E}_v$, where $\mathrm{id}_{\mathbb{N}}$ is the identity function on $\mathbb{N}$.

HINT: Use that each $x \in \mathbb{R}$ can be uniquely written as
$$x = \frac{a_1}{1!} + \frac{a_2}{2!} + \cdots + \frac{a_n}{n!} + \cdots ,$$
where $a_1 = \lfloor x \rfloor$, for each $n \geqslant 2$, $a_n \in \{0, 1, ..., n-1\}$ and $\exists^\infty n(a_n \neq n-1)$; the latter condition is to ensure uniqueness.

91. Let E be an equivalence relation on a Polish space X. Prove that $\mathrm{id}(2^{\mathbb{N}}) \leqslant_B E$ iff $\mathrm{id}(2^{\mathbb{N}}) \sqsubseteq_B E$ iff $\mathrm{id}(2^{\mathbb{N}}) \sqsubseteq_c E$.

92. Fill in the details in the proof of Mycielski's theorem; namely, given a meager equivalence relation E on a Polish space X, write $E = \bigcup_n F_n$, where F_n are increasing and nowhere dense, and construct a Cantor scheme $(U_s)_{s \in 2^{<\mathbb{N}}} \subseteq X$ of vanishing diameter (with respect to a fixed complete metric d for X) with the following properties:

(i) U_s is nonempty open, for each $s \in 2^{<\mathbb{N}}$;

(ii) $\overline{U_{s^\frown i}} \subseteq U_s$, for each $s \in 2^{<\mathbb{N}}$, $i \in \{0, 1\}$;

(iii) $(U_s \times U_t) \cap F_n = \emptyset$, for all distinct $s, t \in 2^n$ and $n \in \mathbb{N}$.

93. Let $(X, \mathcal{T})$ be a Polish space and let E be an equivalence relation on X. For a family $\mathcal{F}$ of subsets of X, we say that $\mathcal{F}$ *generates* E if
$$xEy \iff \forall A \in \mathcal{F}(x \in A \Leftrightarrow y \in A).$$
Prove that the following are equivalent:

(1) E is smooth;

(2) There is a Polish topology $\mathcal{T}_E \supseteq \mathcal{T}$ on X (and hence automatically $\mathcal{B}(\mathcal{T}_E) = \mathcal{B}(\mathcal{T})$) such that E is closed in $(X^2, \mathcal{T}_E^2)$.

CAUTION: It is easy to make E closed in X^2 by refining the topology of X^2, but here we have to refine the topology of X so that E becomes closed in X^2.

(3) E is generated by a countable Borel family $\mathcal{F} \subseteq \mathcal{B}(\mathcal{T})$.

HINT: For (1)$\Rightarrow$(2), consider a Borel function witnessing the smoothness of E and make it continuous. For (2)$\Rightarrow$(3), assuming that E is closed, write $X^2 \setminus E = \bigcup_n U_n \times V_n$, with U_n, V_n disjoint open, and note that the saturations $[U_n]_E$ and $[V_n]_E$ are disjoint analytic sets; separate them by an invariant Borel set.

94. (Blackwell's theorem) Let X be a Polish space and E be an equivalence relation on X generated by a countable family $\{B_n\}_{n \in \mathbb{N}}$ of Borel sets. Prove that a Borel set $B \subseteq X$ is E-invariant iff it belongs to the σ-algebra generated by $\{B_n\}_{n \in \mathbb{N}}$.

Hint: For $\Rightarrow$ direction, consider the function $f : X \to 2^{\mathbb{N}}$ by $x \mapsto (x_n)_{n\in\mathbb{N}}$, where $x_n = 1 \Leftrightarrow x \in B_n$, and use Question 70.

95. **Prisoners and hats** ($\mathbb{E}_0$ version). This question illustrates the nonsmoothness of $\mathbb{E}_0$, more particularly, how having a selector for $\mathbb{E}_0$ (provided by AC) causes unintuitive things.

Problem. ω-many prisoners are sentenced to death, but they can get out under the following condition. On the day of the execution they will be lined up, i.e., enumerated $(p_n)_{n\in\mathbb{N}}$, so that everybody can see everybody else but themselves. Each of the prisoners will have a red or blue hat put on them, but he/she won't be told which color it is (although they can see the other prisoners' hats). On command, all the prisoners at once make a guess as to what color they think their hat is. If all but finitely many prisoners guess correctly, they all go home free; otherwise all of them are executed. The good news is that the prisoners think of a plan the day before the execution, and indeed, all but finitely many prisoners guess correctly the next day, so everyone is saved. How do they do it?

96. For Polish spaces X, Y, a function $f : X \to Y$ is called *universally measurable* if the f-preimages of open sets in Y are universally measurable sets. Prove that the composition of two universally measurable functions is universally measurable.

97. For a Borel equivalence relation E, show that if there is universally measurable reduction of E to $\mathrm{Id}(2^{\mathbb{N}})$, then E is smooth (i.e. there is a *Borel* reduction of E to $\mathrm{Id}(2^{\mathbb{N}})$).

Hint: It's ok to use big theorems.

98. Let $S \subseteq 2^{<\mathbb{N}}$.

(a) If S contains at most one element of each length, then $\mathcal{G}_S$ is acyclic[15].

Hint: Suppose there is a cycle (with no repeating vertex) and consider the longest $s \in S$ associated with its edges.

(b) If S contains at least one element of each length, then $E_{\mathcal{G}_S} = \mathbb{E}_0$.

Hint: For each $n \in \mathbb{N}$, show by induction on n that for each $s, t \in 2^n$ and $x \in 2^{\mathbb{N}}$, there is a path in $\mathcal{G}_S$ from $s^\frown x$ to $t^\frown x$, i.e. $s^\frown x$ can be transformed to $t^\frown x$ by a series of appropriate bit flips.

99. **Prisoners and hats** (Hamming graph version[16]). This question illustrates that the *Hamming graph* H on $2^{\mathbb{N}}$ does not admit a reasonable 2-coloring. The Hamming graph H is defined by putting an edge between two binary sequences if they differ by exactly one bit. Thus, H is a cousin of G_0 and $E_H = \mathbb{E}_0$.

Problem. ω-many prisoners are sentenced to death, but they can get out under the following condition. On the day of the execution they will be lined up, i.e., enumerated $(p_n)_{n\in\mathbb{N}}$, so that everybody can see everybody else but themselves. Each of the prisoners will have a red or blue hat put on them, but he/she won't be told which color it is

[15] Here we treat $\mathcal{G}_S$ as an undirected graph, i.e. we consider edges (x, y) and (y, x) to be the same.

[16] Thanks to Dat P. Nguyen for coming up with this version of the problem.

(although they can see the other prisoners' hats). On command, each prisoner, one-by-one (starting from p_0, then p_1, then p_2, etc.), makes a guess as to what color they think their hat is. Whoever guesses right, goes home free. The good news is that the prisoners think of a plan the day before the execution, so that at most one prisoner is executed. How do they do it?

100. Let E, F be countable Borel equivalence relations on standard Borel spaces X, Y, respectively, and let $A \subseteq X$ be a Borel E-*complete section*, i.e., it meets every E-class.

(a) Construct a Borel reduction $\pi : E \to E|_A$ whose graph is contained in E, i.e., $\pi(x) E x$ for each $x \in X$.

Hint: Luzin–Novikov (what else).

(b) Deduce that any Borel reduction $f_A : E|_A \to F$ extends to a Borel reduction $f : E \to F$.

101. [17] Prove the following more general version of the Schröder–Bernstein theorem for equivalence relations.

Theorem. *Let E, F be countable Borel equivalence relations on standard Borel spaces X, Y, respectively. If $E \leqslant_B F$ and $F \leqslant_B E$, then there is a Borel reduction $f : E \to F$ that descends to a bijection between X/E and Y/F, i.e., $[f(X)]_F = Y$.*

Hint: Because reductions descend to injections between X/E and Y/F, the Schröder–Bernstein algorithm applied to these injections would produce a bijection between X/E and Y/F. Mimic on X and Y what happens on the level of quotients by taking the saturations of all of the sets that show up in the Schröder–Bernstein algorithm. This will produce a required reduction to F of a restriction of E to some Borel E-complete section. Now apply Problem 100.

102. Projections along graphs. Let $\mathcal{G}$ be a locally countable Borel graph on a standard Borel space X and let $A \subseteq X$ be a Borel $E_{\mathcal{G}}$-complete section. A *projection onto A along $\mathcal{G}$* is a function $\pi : X \to A$ that maps each point in X to a $\mathcal{G}$-closest (in the graph distance) element of A (in particular, $\pi|_A = \mathrm{id}_A$). Call this projection π *coherent* if, moreover, for each $x \in X$, $\pi^{-1}(\pi(x))$ contains a shortest path from x to $\pi(x)$ (thus all points on that path are mapped to the same point $\pi(x)$). Prove that there is a coherent Borel projection onto A along $\mathcal{G}$. Deduce part (a) of Problem 100 from this.

Hint: Luzin–Novikov assigns natural numbers to the edges of $\mathcal{G}$, providing a lexicographic ordering of the paths in $\mathcal{G}$.

103. Let $\mathcal{G}$ be a directed Borel graph on a Polish space $(X, \mathcal{T})$. For $A \subseteq X$, let

$$N^o_{\mathcal{G}}(A) = \{x \in X : \exists y \in A \text{ with } (x, y) \in \mathcal{G}\}$$

denote the set of out-neighbors of vertices in A in $\mathcal{G}$. If $A = \{x\}$, we just write $N^o_{\mathcal{G}}(x)$. Put $d^o_{\mathcal{G}}(x) := |N^o_{\mathcal{G}}(x)|$ and call it the out-degree of x in $\mathcal{G}$. Lastly, if $\mathcal{G}$ is undirected, i.e. symmetric, then we drop o from the superscripts and simply write $N_{\mathcal{G}}$ and $d_{\mathcal{G}}$.

[17] Thanks to Ruiyuan (Ronnie) Chen for pointing out this statement.

(a) Suppose that the out-degree of each vertex is countable and prove that $\chi_{\mathcal{B}}(\mathcal{G}) \leqslant \aleph_0$ iff there is a Polish topology $\mathcal{T}_0 \supseteq \mathcal{T}$ such that for every $x \in X$, $x \notin \overline{N_{\mathcal{G}}(x)}^{\mathcal{T}_0}$.

HINT: For $\Leftarrow$, use the fact that $x \notin \overline{N_{\mathcal{G}}(x)}^{\mathcal{T}_0}$ is witnessed by a basic open set $U_n \in \mathcal{T}_0$.

(b) Conclude if the out-degree of each vertex is finite, then $\chi_{\mathcal{B}}(\mathcal{G}) \leqslant \aleph_0$.

104. Show that if an undirected locally countable[18] Borel graph $\mathcal{G}$ has a countable Borel chromatic number, then it admits a Borel maximal $\mathcal{G}$-independent set[19] $I \subseteq X$, i.e. a maximal $\mathcal{G}$-independent set that happens to be Borel.

105. Let $\mathcal{G}$ be a Borel graph on a standard Borel space X.

(a) Prove that if the maximum degree of $\mathcal{G}$ is $\leqslant d$ (i.e. $d_{\mathcal{G}}(x) \leqslant d$ for each $x \in X$), then $\chi_{\mathcal{B}}(\mathcal{G}) \leqslant d + 1$.

HINT: Prove by induction[20] on d using Question 104.

(b) Conclude that the Borel chromatic number of the graph induced by an irrational rotation of the unit circle is 3 and define a Borel 3-coloring of this graph more explicitly (by drawing a picture) such that each color is a finite union of half-open arcs.

106. This is a proof of the Feldman–Moore theorem in terms of Borel edge-colorings.

For a set X, we refer to an pair $(x, y) \in X^2$ as a *directed edge* with *source* x and *target* y. We say that edges (x, y) and $-(x, y) := (y, x)$ are *parallel*. Directed edges $(x, y), (x', y')$ are said to be *source-incident* (resp., *target-incident*, *mixed-incident*) if $x = x'$ (resp., $y = y'$, $x = y'$ or $y = x'$). We say that they are *whatsoever-incident* if they are incident in one of the three aforementioned ways.

Let E be a countable Borel equivalence relation on a Polish space X.

(a) By the Luzin–Novikov theorem, $G := E \setminus \mathrm{Id}_X = \bigcup_n f_n$, where $f_n : X \rightharpoonup X$ is a Borel partial function. This defines $c_0 : G \to \mathbb{N}$ by $(x, y) \mapsto$ the least $n \in \mathbb{N}$ such that $f_n(x) = y$. Show that c_0 is Borel and that for any distinct source-incident edges $e, e' \in G$, $c_0(e) \neq c(e')$. Hence, for any target-incident edges $e, e' \in G$, $c_0(-e) \neq c_0(-e')$.

(b) Because X is second countable, we can write $X^2 \setminus \mathrm{Id}_X = \bigcup_m (U_m \times V_m)$, where $U_m, V_m \subseteq X$ are open and $U_m \cap V_m = \emptyset$. This defines $c_1 : G \to \mathbb{N}$ by $(x, y) \mapsto$ the least $m \in \mathbb{N}$ such that $(x, y) \in U_m \times V_m$. Show that c_1 is Borel and that for any mixed-incident edges $e, e' \in G$, $c_1(e) \neq c_1(e')$.

(c) Conclude that $c : G \to \mathbb{N}^3$ defined by $e \mapsto (c_0(e), c_0(-e), c_1(e))$ is a *directed edge-coloring* of G, in the strong sense that any two whatsoever-incident edges get

[18] Every vertex has only countably-many neighbors.

[19] A set $I \subseteq X$ is $\mathcal{G}$-*independent* if it doesn't contain any $\mathcal{G}$-adjacent vertices, i.e. $I^2 \cap \mathcal{G} = \emptyset$. Such a set is called *maximal* if it is not a proper subset of another $\mathcal{G}$-independent set.

[20] Thanks to Wei Dai for suggesting a proof by induction.

different colors. Thus, G admits a Borel directed edge-coloring with countable-many colors.

(d) Show that for any Borel directed edge-coloring $c : G \to \mathbb{N}$, the map $c' : G \to \mathbb{N}$ defined by $e \mapsto \min\{c(e), c(-e)\}$ is a Borel (*undirected*) *edge-coloring* of G, in the sense that the color of an undirected edge is well-defined (i.e., parallel directed edges get the same color) and incident undirected edges get different colors (i.e., nonparallel whatsoever-incident directed edges get different colors).

107. Let X be a standard Borel space and $T : X \to X$ be a Borel transformation on X (not necessarily countable to one). Letting $G \coloneqq \mathrm{graph}(T)$.

(a) Prove that $\chi_{\mathcal{B}}(\mathrm{graph}(T)) \in \{1, 2, 3, \aleph_0\}$.

HINT: $\chi_{\mathcal{B}}(G) \leqslant \aleph_0$ follows from Question 103. To show that if $\chi_{\mathcal{B}}(G) < \aleph_0$ then $\chi_{\mathcal{B}}(G) \leqslant 3$, note that for each $x \in X$ one of the final colors has to be T-recurrent (i.e. for infinitely-many $n \in \mathbb{N}$, $T^n(x)$ is of that color) by the pigeonhole principle. Restricting to each part where the least such color is the same, let's say color 0, work backwards from this color in the direction of T^{-1}, alternatively assigning to odd and even distances colors 1 and 2.

(b) Conclude that if T is d-to-1 for some fixed $d \in \mathbb{N}$, then $\chi_{\mathcal{B}}(G) \leqslant 3$.

108. Let $\mathcal{G}$ be the aperiodic part of the shift graph on $\mathcal{C}$, i.e., the graph of the left-shift function $s : \mathcal{C} \to \mathcal{C}$, defined by $(x_n) \mapsto (x_{n+1})$ for each aperiodic sequence $(x_n) \in \mathcal{C}$. Prove that the Baire measurable chromatic number of $\mathcal{G}$ is greater than 2. Yet, conclude from part (b) of Question 107 that $\chi_{\mathcal{B}}(G) = 3$.

109. Show that there exists a *universal analytic equivalence relation*, i.e. an analytic equivalence relation $\mathbb{E}_{\boldsymbol{\Sigma}}$ such that any other such equivalence relation is Borel reducible to $\mathbb{E}_{\boldsymbol{\Sigma}}$.

HINT: Take a $\mathcal{C}$-universal set $U \subseteq \mathcal{C} \times \mathcal{N}^2$ for $\boldsymbol{\Sigma}^1_1(\mathcal{N}^2)$ and let $\tilde{U}$ be obtained from U by replacing the fibers U_x, $x \in \mathcal{C}$, with their symmetric and transitive closures, so that each fiber $\tilde{U}_x$ is an equivalence relation on $\mathcal{N}$. Now define an appropriate equivalence relation $\mathbb{E}_{\boldsymbol{\Sigma}}$ on $\mathcal{C} \times \mathcal{N}$.

110. The goal of this question is to show that there is a *universal countable Borel equivalence relation*, i.e., a countable Borel equivalence relation $\mathbb{E}_\infty$ such that any other such equivalence relation is Borel reducible to $\mathbb{E}_\infty$.

(a) Letting $\mathbb{F}_\omega$ be the free group on ω-many generators and Γ be any countable group, define a Borel reduction $\rho : (2^{\mathbb{N}})^\Gamma \to (2^{\mathbb{N}})^{\mathbb{F}_\omega}$ of the orbit equivalence relation E_Γ to the orbit equivalence relation $E_{\mathbb{F}_\omega}$ of the shift actions $\Gamma \curvearrowright (2^{\mathbb{N}})^\Gamma$ and $\mathbb{F}_\omega \curvearrowright (2^{\mathbb{N}})^{\mathbb{F}_\omega}$, respectively.

HINT: Every countable group is a homomorphic image of $\mathbb{F}_\omega$.

(b) Using the Feldman–Moore theorem in tandem with Question 89, conclude that the orbit equivalence relation $E_{\mathbb{F}_\omega}$ of the shift action of $\mathbb{F}_\omega$ on $(2^{\mathbb{N}})^{\mathbb{F}_\omega}$ is a universal countable Borel equivalence relation.