DESCRIPTIVE SET THEORISTS TOO CAN OMIT TYPES

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The Omitting Types theorem is a standard pocket tool in model theory, see, for example, [TZ12, 10.2]. The proofs of this theorem known to the author, including those that use Baire Category theorem on the type space, involve, perhaps implicitly, some version of Henkin's construction. This note gives a short proof of this theorem using only very basic descriptive set theory.

Fix a countable first-order relational language $\mathcal{L}$ and consider all of the first-order notions below in this language.

Let $\mathrm{Mod}_{\mathcal{L}}$ denote the space of all $\mathcal{L}$-structures on $\mathbb{N}$, which we encode as the set $\prod_{R\in\mathcal{L}} 2^{\mathbb{N}^{\mathfrak{a}(R)}}$, where $\mathfrak{a}(R)$ is the arity of R. The latter is a homeomorphic copy of the Cantor space in the product topology $\mathcal{T}$; in particular, it is compact metrizable and hence Polish.

For each (extended) $\mathcal{L}$-formula $\varphi(\vec{x})$ and $\vec{a} \in \mathbb{N}^{|\vec{x}|}$, we denote by $\varphi(\vec{a}/\vec{x})$ the result of the substitution in φ of $\vec{a}$ for $\vec{x}$ and we put

$$\langle \varphi(\vec{a}/\vec{x}) \rangle := \{ \boldsymbol{M} \in \mathrm{Mod}_{\mathcal{L}} : \boldsymbol{M} \models \varphi(\vec{a}/\vec{x}) \}.$$

Let $\mathcal{D}_{\mathcal{L}}$ denote the collection of all such sets.

Lemma 1. *The topology $\overline{\mathcal{T}}$ generated by $\mathcal{D}_{\mathcal{L}}$ is Polish.*

Proof. Enumerate the formulas (φ_n) so that the subformulas of each φ_n come before φ_n. For each $n \in \mathbb{N}$, let $\mathcal{T}_n$ denote the topology generated by $\mathcal{T}$ and the sets of the form $\langle \varphi_k(\vec{a}/\vec{x}) \rangle$ for $k < n$, vector of variables $\vec{x}$ that includes the free variables of φ_k, and $\vec{a} \in \mathbb{N}^{|\vec{x}|}$. By [Kec95, 13.3], it is enough to show that each $\mathcal{T}_n$ is Polish, which we do by induction on n. Noting that $\mathcal{T}_0 = \mathcal{T}$ and is hence Polish, we suppose that $\mathcal{T}_n$ is Polish and prove that $\mathcal{T}_{n+1}$ is Polish. By [Kec95, 13.3] again, it is enough to show that the topology $\mathcal{T}_n'$ generated by $\mathcal{T}_n$ and the set $\langle \varphi_n(\vec{a}/\vec{x}) \rangle$ is Polish for fixed vector of variables $\vec{x}$ that includes the free variables of φ_n and $\vec{a} \in \mathbb{N}^{|\vec{x}|}$.

To this end, we consider cases corresponding to the recursive definition of φ_n. Observing that

- φ_n is atomic $\implies \langle \varphi_n(\vec{a}/\vec{x}) \rangle$ is clopen in $\mathcal{T} \subseteq \mathcal{T}_n$;
- $\varphi_n = \varphi_k \vee \varphi_\ell$ for some $k, \ell < n \implies \langle \varphi_n(\vec{a}/\vec{x}) \rangle = \langle \varphi_k(\vec{a}/\vec{x}) \rangle \cup \langle \varphi_\ell(\vec{a}/\vec{x}) \rangle \in \mathcal{T}_n$;
- $\varphi_n = \exists y \varphi_k$ for some $k < n \implies \langle \varphi_n(\vec{a}/\vec{x}) \rangle = \bigcup_{b\in\mathbb{N}} \langle \varphi_k(\vec{a}, b/\vec{x}, y) \rangle \in \mathcal{T}_n$;

we see that in all these cases $\mathcal{T}_n' = \mathcal{T}_n$ so we are done. In the remaining case $\varphi_n = \neg\varphi_k$ for some $k < n$, $\langle \varphi_n(\vec{a}/\vec{x}) \rangle = \langle \varphi_k(\vec{a}/\vec{x}) \rangle^c$, so $\mathcal{T}_n'$ is Polish by [Kec95, 13.2]. $\square$

For a partial type $\Sigma(\vec{x})$ and $\vec{a} \in \mathbb{N}^{|\vec{x}|}$, put

$$[\Sigma(\vec{a}/\vec{x})] := \{ \boldsymbol{M} \in \mathrm{Mod}_{\mathcal{L}} : \boldsymbol{M} \models \Sigma(\vec{a}/\vec{x}) \} = \bigcap_{\varphi\in\Sigma} \langle \varphi(\vec{a}/\vec{x}) \rangle,$$

so $[\Sigma(\vec{a}/\vec{x})]$ is closed in $\overline{\mathcal{T}}$. For a theory T, let $\mathrm{Mod}_{\mathcal{L}}(T)$ stand for $[T]$; in particular, it is a Polish space in the relative topology of $\overline{\mathcal{T}}$. We write $\langle \varphi(\vec{a}/\vec{x}) \rangle_T$ and $[\Sigma(\vec{a}/\vec{x})]_T$ for the intersections of the corresponding sets $\langle \varphi(\vec{a}/\vec{x}) \rangle$ and $[\Sigma(\vec{a}/\vec{x})]$ with the closed set $\mathrm{Mod}_{\mathcal{L}}(T)$.

Lemma 2. *For a theory T and a partial type $\Sigma(\vec{x})$, the following are equivalent:*

(1) *$\Sigma(\vec{x})$ is isolated in T.*

(2) *There is a formula $\varphi(\vec{x})$ such that for all $\vec{a} \in \mathbb{N}^{|\vec{x}|}$, $[\Sigma(\vec{a}/\vec{x})] \supseteq \langle \varphi(\vec{a}/\vec{x}) \rangle_T \neq \emptyset$.*

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The title of this note is, of course, in reference to the quote in [Sac10, Par. 1 of Sec. 18].

(3) *For some* $\vec{a}_0 \in \mathbb{N}^{|\vec{x}|}$, $[\Sigma(\vec{a}_0/\vec{x})]_T$ *has nonempty interior in* $\overline{\mathcal{T}}$.

Proof. The implications (1)⇔(2)⇒(3) are clear, so we prove (3)⇒(2). Supposing (3), $[\Sigma(\vec{a}_0/\vec{x})]_T$ contains a basic open set of the form $\left\langle \psi(\vec{a}_0, \vec{b}_0/\vec{x}, \vec{y}) \right\rangle_T \neq \emptyset$, where $\vec{x}$ and $\vec{y}$, as well as $\vec{a}_0$ and $\vec{b}_0$, are disjoint.

Claim. $[\Sigma(\vec{a}_0/\vec{x})]_T \supseteq \varphi(\vec{a}_0/\vec{x})$ for some formula $\varphi(\vec{x})$.

Proof of Claim. Because any permutation $\pi \in S_\infty$ that fixes $\vec{a}_0$ pointwise fixes $[\Sigma(\vec{a}_0/\vec{x})]_T$ setwise and takes $\left\langle \psi(\vec{a}_0, \vec{b}_0/\vec{x}, \vec{y}) \right\rangle_T$ to $\left\langle \psi(\vec{a}_0, \pi(\vec{b}_0)/\vec{x}, \vec{y}) \right\rangle_T$, we may apply permutations that map $\vec{b}_0$ to any $\vec{b} \in \mathbb{N}^{|\vec{y}|}$ disjoint from $\vec{a}_0$ and get that

$$[\Sigma(\vec{a}_0/\vec{x})]_T \supseteq \bigcup_{\vec{b} \in \mathbb{N}^{|\vec{y}|}} \left\langle \psi(\vec{a}_0, \vec{b}/\vec{x}, \vec{y}) \right\rangle_T .$$

Thus, $[\Sigma(\vec{a}_0/\vec{x})]_T \supseteq \langle \varphi(\vec{a}_0/\vec{x}) \rangle_T$, where $\varphi := \exists \vec{y} (\psi \wedge$ "$\vec{x}$ and $\vec{y}$ are disjoint"). ⊠

It remains to show that this is true for all $\vec{a} \in \mathbb{N}^{|\vec{x}|}$. But this again follows by applying permutations $\mathbb{N}$ that take $\vec{a}_0$ to any $\vec{a} \in \mathbb{N}^{|\vec{x}|}$ because they map $[\Sigma(\vec{a}_0)]_T$ to $[\Sigma(\vec{a})]_T$ and $\langle \varphi(\vec{a}_0/\vec{x}) \rangle_T$ to $\langle \varphi(\vec{a}/\vec{x}) \rangle_T$. □

For a partial type $\Sigma(\vec{x})$, let $\mathrm{Omit}_T \Sigma(\vec{x})$ denote the set of all models $\boldsymbol{M} \in \mathrm{Mod}_{\mathcal{L}}(T)$ that omit it:

$$\mathrm{Omit}_T \Sigma(\vec{x}) := \bigcap_{\vec{a} \in \mathbb{N}^{|\vec{x}|}} [\Sigma(\vec{a}/\vec{x})]_T^c .$$

Theorem 3. *For any* $\mathcal{L}$*-theory* T *and any countable set* $\{\Sigma(\vec{x}_n)\}_{n \in \mathbb{N}}$ *of nonisolated partial* $\mathcal{L}$*-types, the set* $\bigcap_n \mathrm{Omit}_T \Sigma_n(\vec{x})$ *is a dense* G_δ *subset of* $\mathrm{Mod}_{\mathcal{L}}(T)$ *in the topology* $\overline{\mathcal{T}}$.

Proof. By Lemma 2, the complement of each $\mathrm{Omit}_T \Sigma_n(\vec{x}_n)$ is a countable union of nowhere dense closed sets, hence meager. Thus, $\bigcap_n \mathrm{Omit}_T \Sigma_n(\vec{x}_n)$ is comeager, so it must be dense by the Baire Category theorem, which applies to $\overline{\mathcal{T}}$ since the latter is Polish by Lemma 1. □

Corollary 4 (Omitting Types)**.** *For any* $\mathcal{L}$*-theory* T *and any countable set* $\{\Sigma(\vec{x}_n)\}_{n \in \mathbb{N}}$ *of partial* $\mathcal{L}$*-types, there is a model of* T *omitting* $\Sigma_n(\vec{x}_n)$ *for each* $n \in \mathbb{N}$.

References

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