# CONVERGENCE IN MEASURE

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Fix a measure space $(X, \mathcal{M}, \mu)$ and everything below is with respect to this space. By a *measurable function*, we mean a function $X \to \overline{\mathbb{R}} \coloneqq [-\infty, +\infty]$ that is $(\mathcal{M}, \mathcal{B}(\overline{\mathbb{R}}))$-measurable.

## 1. Parameterized difference sets and quasi-metrics

For measurable functions $f, g$ and $\alpha \geqslant 0$, put

$$\begin{aligned}\Delta_\alpha(f, g) &\coloneqq \{x \in X : |f(x) - g(x)| > \alpha\} \\ \delta_\alpha(f, g) &\coloneqq \mu(\Delta_\alpha(f, g)).\end{aligned}$$

**Observation 1** (Decreasing monotonicity)**.** *For $\alpha, \beta \geqslant 0$ and measurable functions $f, g$,*

$$\alpha \geqslant \beta \implies \Delta_\alpha(f, g) \subseteq \Delta_\beta(f, g) \implies \delta_\alpha(f, g) \leqslant \delta_\beta(f, g)$$

For each $\alpha \geqslant 0$, $\delta_\alpha$ is a metric-like function on the space of all measurable functions, but it is not even a pseudo-metric because the triangle inequality easily fails: indeed, take $f, g, h$ to be constant functions $0, 2, 4$, respectively. Then $\delta_3(f, h) = \mu(X)$, whereas $\delta_2(f, g) = \delta_2(g, h) = 0$. However, changing the parameter $\alpha$, gives the following substitute for the triangle inequality.

**Lemma 2** (Quasi-$\triangle$-inequality)**.** *Let $f, g, h$ be measurable functions and let $\alpha \geqslant 0$. For any partition $\alpha = \beta + \gamma$ into nonnegative reals,*

(2.a) $\Delta_\alpha(f, h) \subseteq \Delta_\beta(f, g) \cup \Delta_\gamma(g, h)$,

(2.b) $\delta_\alpha(f, h) \leqslant \delta_\beta(f, g) + \delta_\gamma(g, h)$.

*Proof.* (2.a) is immediate from the usual triangle inequality for reals and it implies (2.b). □

**Observation 3.** *For any measurable functions $f, g$ and decreasing positive sequence $(\alpha_k)_k \downarrow 0$,*

$$\Delta_0(f, g) = \bigcup_{k \in \mathbb{N}}^{\uparrow} \Delta_{\alpha_k}(f, g).$$

## 2. Convergence in measure

**Definition 4.** For a sequence $(f_n)_n$ of measurable functions and a measurable function $f$, say that $(f_n)_n$ *converges to $f$ in measure*, and write $f_n \to_\mu f$, if for each parameter $\alpha > 0$,

$$\delta_\alpha(f_n, f) \to 0 \text{ as } n \to \infty.$$

Note that the definition of convergence in measure only involves positive $\alpha$, excluding $\alpha = 0$, and it is not true in general that it implies $\delta_0(f_n, f) \to 0$. Indeed, for any measure space $(X, \mathcal{M}, \mu)$, take $f \equiv 0$ and $f_n \equiv n^{-1}$.

Call two measurable functions $f, g$ *$\mu$-distinct* if even $\mu$ sees that they are distinct, i.e. $d_0(f, g) > 0$. In other words, if $f \nsim_\mu g$, where $\sim_\mu$ is the relation of being equal a.e.

**Proposition 5** (Hausdorfness of the topology)**.** *For any two $\mu$-distinct measurable functions $f, g$, there is $\alpha > 0$ such that $\delta_\alpha(f, g) > 0$.*

*Proof.* Follows from Observation 3 and the countable additivity of $\mu$. □

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**Corollary 6.** *If $f_n \to_\mu f$ and $f_n \to_\mu f'$, then $f = f'$ a.e.*

*Proof.* For each fixed $\alpha > 0$, by the quasi-$\triangle$-inequality,

$$\delta_\alpha(f, f') \leqslant \delta_{\frac{\alpha}{2}}(f, f_n) + \delta_{\frac{\alpha}{2}}(f_n, f') \to 0 \text{ as } n \to \infty,$$

so $\delta_\alpha(f, f') = 0$, and hence $\delta_0(f, f') = 0$, by Observation 3. ☐

## 3. Cauchy in measure

**Definition 7.** Call a sequence $(f_n)_n$ of measurable functions *Cauchy in measure* if for each parameter $\alpha > 0$, $\delta_\alpha(f_n, f_m) \to 0$ as $n, m \to \infty$.

**Proposition 8** (Convergence $\Rightarrow$ Cauchy)**.** *For measurable functions $f$ and $(f_n)_n$, if $f_n \to_\mu f$ then $(f_n)_n$ is Cauchy.*

*Proof.* For given $\alpha, \varepsilon > 0$, fix $N \in \mathbb{N}$ large enough so that $\delta_{\frac{\alpha}{2}}(f_n, f) < \frac{\varepsilon}{2}$, so, by the quasi-$\triangle$-inequality, for all $n, m \geqslant N$, $\delta_\alpha(f_n, f_m) \leqslant \delta_{\frac{\alpha}{2}}(f_n, f) + \delta_{\frac{\alpha}{2}}(f, f_m) < \varepsilon$. ☐

**Lemma 9.** *Let $(f_n)_n$ be a Cauchy in measure sequence of measurable functions.*

(9.a) *For any sequences $(\alpha_k)_k$ and $(\varepsilon_k)_k$ of positive reals, there is a subsequence $(g_k)_k := (f_{n_k})_k$ such that for every $k \in \mathbb{N}$, $\delta_{\alpha_k}(g_{k+l}, g_{k+m}) < \varepsilon_k$ for all $l, m \in \mathbb{N}$.*

(9.b) *If a subsequence $(f_{n_k})_k$ converges in measure to a measurable function $f$, then $f_n \to_\mu f$.*

*Proof.* (9.a): For each $\alpha_k$ and $\varepsilon_k$, the Cauchy condition gives $n_k \in \mathbb{N}$ such that for all $\ell, m \geqslant n_k$, $\delta_{\alpha_k}(f_\ell, f_m) < \varepsilon_k$. Since we can always take these $n_k$ to be increasing, we get a subsequence $(g_k)_k := (f_{n_k})_k$ with the desired property.

(9.b): Fix $\alpha > 0$ and $\varepsilon > 0$, so the Cauchy condition gives $N$ such that for any $n, m \geqslant N$, $\delta_{\frac{\alpha}{2}}(f_n, f_m) < \frac{\varepsilon}{2}$. Also, the convergence $f_{n_k} \to_\mu f$ gives $K \in \mathbb{N}$, such that for any $k \geqslant K$, $\delta_{\frac{\alpha}{2}}(f_{n_k}, f) < \frac{\varepsilon}{2}$. Thus, for any $n \geqslant N$, taking $k \geqslant \max\{N, K\}$, the quasi-$\triangle$-inequality implies

$$\delta_\alpha(f_n, f) \leqslant \delta_{\frac{\alpha}{2}}(f_n, f_{n_k}) + \delta_{\frac{\alpha}{2}}(f_{n_k}, f) < \varepsilon.$$ ☐

*Notation* 10. For any summable sequence $(\beta_k)_k$ of positive reals and for each $K \in \mathbb{N}$, put $\bar{\beta}_K := \sum_{k \geqslant K} \beta_k$, so $\bar{\beta}_K \to 0$ as $K \to \infty$.

**Theorem 11** (Cauchy $\Rightarrow$ convergence)**.** *If a sequence $(f_n)_n$ of measurable functions is Cauchy in measure, then $f_n \to_\mu f$ for some $\overline{M}$-measurable function $f$. Moreover, there is a subsequence $(f_{n_k})_k$ with $f_{n_k} \to f$ a.e.*

*Proof.* Applying (9.a) to any positive summable sequences $(\alpha_k)_k$ and $(\varepsilon_k)_k$, we get a subsequence $(g_k)_k$ such that for each $k$, $\delta_{\alpha_k}(g_k, g_{k+1}) < \varepsilon_k$. Thinking of the $\Delta_{\alpha_k}(g_k, g_{k+1})$ as the bad sets, denote them by $B_k$, so $\mu(B_k) < \varepsilon_k$.

*Claim* 12. For any $K \in \mathbb{N}$ and any $x \notin \bigcup_{k \geqslant K} B_k$, $|g_K(x) - g_{K+m}(x)| < \bar{\alpha}_K$ for all $m \in \mathbb{N}$.

*Proof of Claim.* By the $\triangle$-inequality for reals, we have

$$|g_K(x) - g_{K+m}(x)| \leqslant \sum_{K \leqslant k < K+m} |g_k(x) - g_{k+1}(x)|.$$

Because $x \notin B_k$ for $k \geqslant K$, it satisfies $|g_k(x) - g_{k+1}(x)| \leqslant \alpha_k$, so the last sum is at most $\sum_{K \leqslant k < K+m} \alpha_k < \bar{\alpha}_K$. ⊠

*Claim* 13. If $x \notin B := \{x \in X : \exists^\infty k \; x \in B_k\} = \bigcap_{K \in \mathbb{N}} \bigcup_{k \geqslant K} B_k$, then $\big(g_k(x)\big)_k$ is Cauchy.

*Proof of Claim.* Fixing $x \notin B$ and $\varepsilon > 0$, there is $K \in \mathbb{N}$ such that $x \notin \bigcup_{k \geqslant K} B_k$ and we may take large enough such $K$ so that $2 \cdot \bar{\alpha}_K < \varepsilon$. Whence, Claim 12 gives

$$|g_{K+l}(x) - g_{K+m}(x)| \leqslant |g_{K+l}(x) - g_K(x)| + |g_K(x) - g_{K+m}(x)| < 2 \cdot \bar{\alpha}_K < \varepsilon. \qquad \boxtimes$$

Luckily, the $B_k$ have summable measures, so the Borel–Cantelli lemma implies that $B$ is null. But then Claim 13 implies that for a.e. $x \in X$, $\big(g_k(x)\big)_k$ is Cauchy, so it has a limit, which we denote by $f(x)$. Because $g_k \to f$ a.e., $f$ is $\overline{M}$-measurable.

*Claim* 14. $g_K \to_\mu f$.

*Proof of Claim.* Fixing $\alpha > 0$, we need to show that $\delta_\alpha(g_K, f) \to 0$. For any $K \in \mathbb{N}$ and $x \notin \bigcup_{k \geqslant K} B_k$, applying Claim 12 and letting $m \to \infty$, we get $|g_K(x) - f(x)| \leqslant \bar{\alpha}_K$, so

$$\Delta_{\bar{\alpha}_K}(g_K, f) \subseteq \bigcup_{k \geqslant K} B_k.$$

Taking $K$ large enough so that $\bar{\alpha}_K \leqslant \alpha$,

$$\delta_\alpha(g_K, f) \leqslant \delta_{\bar{\alpha}_K}(g_K, f) \leqslant \mu\left(\bigcup_{k \geqslant K} B_k\right) < \sum_{k \geqslant K} \varepsilon_k = \bar{\varepsilon}_K \to 0 \text{ as } K \to \infty. \qquad \boxtimes$$

Lastly, (9.b) implies that $f_n \to_\mu f$. □

## 4. Relationship with other modes of convergence

**Corollary 15.** *For measurable functions $f$ and $(f_n)_n$, if $f_n \to_\mu f$ then there is a subsequence $(f_{n_k})_k$ with $f_{n_k} \to f$ a.e.*

*Proof.* Follows from Proposition 8 and Theorem 11. □

**Proposition 16.** *If $f_n \to_{L^1} f$ then $f_n \to_\mu f$.*

*Proof.* For each $\alpha > 0$, $\alpha \cdot \mathbb{1}_{\Delta_\alpha(f_n, f)} < |f_n - f| \cdot \mathbb{1}_{\Delta_\alpha(f_n, f)}$, so

$$\alpha \cdot \delta_\alpha(f_n, f) = \int \alpha \cdot \mathbb{1}_{\Delta_\alpha(f_n, f)} \leqslant \int |f_n - f| \cdot \mathbb{1}_{\Delta_\alpha(f_n, f)} \leqslant \|f_n - f\|_1,$$

whence $\delta_\alpha(f_n, f) \leqslant \frac{1}{\alpha}\|f_n - f\|_1 \to 0$ as $n \to \infty$. □

**Corollary 17** (Summary). $f_n \to_{L^1} f \implies f_n \to_\mu f \implies \exists (f_{n_k})_k,\ f_{n_k} \to f$ *a.e.*

The following examples show that all of the other possible implications are false in general.

**Examples.**

**(18.a)** $\mathbb{1}_{[n,n+1)} \to 0$ pointwise, but not in measure, and hence also not in $L^1$.

**(18.b)** $n\mathbb{1}_{(0,\frac{1}{n})} \to 0$ both pointwise and in measure, but not in $L^1$.

**(18.c)** Let $(g_m)_{m \in \mathbb{N}} := (f_n^{(k)})_{n \geqslant 1}^{k<n}$ be the typewritter sequence, i.e. $f_n^{(k)} := \mathbb{1}_{([\frac{k}{n}, \frac{k+1}{n}])}$. Then $g_m \to 0$ in $L^1$, hence also in measure, but the sequence $\big(g_m(x)\big)_{m \in \mathbb{N}}$ diverges for every $x \in [0, 1]$.