BAIRE-MEASURABLE SETS

ANUSH TSERUNYAN

Observation 1. *For any sets A, B, C,*

(1.a) *Involution:* $A \triangle A = \emptyset$.

(1.b) *Identity:* $A \triangle \emptyset = A$.

(1.c) *Associativity:* $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

(1.d) $A \triangle B = A^c \triangle B^c$.

Let X be a metric (topological) space.

Definition 2. A set $A \subseteq X$ is called *Baire-measurable* if $A = U \triangle M$ for some open set U and a meager set M.

Lemma 3. *A set $A \subseteq X$ is Baire-measurable if and only if $A \triangle U$ is meager for some open set $U \subseteq X$.*

Proof. Follows from the equivalence

$$A = U \triangle M \iff U \triangle A = M,$$

which is obtained by taking the symmetric difference with U on both sides and using (1.a), (1.b), and (1.c). $\square$

Below, we will be using Lemma 3 as the definition of Baire-measurable.

Lemma 4. *Baire-measurable subsets of X are closed under countable unions.*

Proof. Suppose that A_n is a Baire-measurable set for each $n \in \mathbb{N}$, and we need to show that $A := \bigcup_{n \in \mathbb{N}} A_n$ is Baire-measurable. By the Baire-measurability of the A_n, there are open sets $U_n \subseteq X$ such that $A_n \triangle U_n$ is meager. Put $U := \bigcup_{n \in \mathbb{N}} U_n$ and it is enough to show that $A \triangle U$ is meager. But observe that

$$A \triangle U = (A \setminus U) \cup (U \setminus A) = \bigcup_{n \in \mathbb{N}} (A_n \setminus U) \;\cup\; \bigcup_{n \in \mathbb{N}} (U_n \setminus A)$$

and the sets $A_n \setminus U$ and $U_n \setminus A$ are meager because they are contained in $A_n \triangle U_n$, so $A \triangle U$ is meager because it is a countable union of meager sets. $\square$

Lemma 5. *Closed subsets of X are Baire-measurable.*

Proof. Let $K \subseteq X$ be a closed subset of a metric space X. Let B be the boundary of K, i.e. $B = \overline{K} \setminus \mathrm{Int}(K)$. But K is closed, so $\overline{K} = K$, and hence, $B = K \setminus \mathrm{Int}(K) \subseteq K$, so $\mathrm{Int}(B) \subseteq \mathrm{Int}(K)$, which implies that $\mathrm{Int}(B) = \emptyset$. We also have that B is closed, being a closed set minus open, so it is nowhere dense. Thus, $K \triangle \mathrm{Int}(K) = K \setminus \mathrm{Int}(K) = B$ is nowhere dense, so K is Baire-measurable. $\square$

Lemma 6. *For sets $A, B \subseteq X$, if A is Baire-measurable and $A \triangle B$ is meager, then B is also Baire-measurable.*

Proof. Suppose that A is Baire-measurable, so there is an open set U such that $A \triangle U$ is meager. But then

$$\begin{aligned} B \triangle U &= B \triangle (\emptyset \triangle U) \\ \text{[by (1.a)]} \quad &= B \triangle \big((A \triangle A) \triangle U\big) \\ \text{[by (1.c)]} \quad &= (B \triangle A) \triangle (A \triangle U) \\ &\subseteq (B \triangle A) \cup (A \triangle U), \end{aligned}$$

and both set $B \triangle A$ and $A \triangle U$ are meager, so $B \triangle U$ is also meager, and hence B is Baire-measurable. $\square$

Proposition 7. *The Baire-measurable subsets of X form a σ-algebra.*

Proof. The emptyset is trivially Baire-measurable and Lemma 4 shows the closure under countable unions, so it remains to show the closure under complements. Fix a Baire-measurable set $A \subseteq X$, so there is an open set $U \subseteq X$ such that $A \triangle U$ is meager. By (1.d), $A^c \triangle U^c = A \triangle U$, so $A^c \triangle U^c$ is also meager. But U^c is closed and hence Baire-measurable by Lemma 5, so A^c is also Baire-measurable by Lemma 6. $\square$