COUNTABLE COMPACT HAUSDORFF SPACES ARE POLISH

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The question of whether countable compact Hausdorff spaces are Polish came up when having beer with a group of set theorists at a bar after an AMS meeting. Although the question perhaps wasn't terribly interesting, I had fun finding the (positive) answer and the current note is the write-up of the proof.

To prove that a countable compact Hausdorff space X is Polish, it is enough to show that it is first-countable: indeed, the countability of X then implies that X is second-countable, allowing us to apply the Urysohn metrization theorem; thus X is compact metrizable and hence Polish. The proof below of the first-countability of X (somewhat surprisingly) goes through showing that X must be zero-dimensional.

Proposition 1. *The cardinality of any compact Hausdorff perfect nonempty topological space X is at least continuum.*

Proof. Using the perfectness and the normality of X, construct a Cantor scheme $(U_s)_{s \in 2^{<\mathbb{N}}}$ such that

(i) U_s is nonempty open;
(ii) $\overline{U_{s^\frown i}} \subseteq U_s$, for $i \in \{0, 1\}$.

For each $x \in \mathcal{C}$, $\bigcap_n U_{x|_n} = \bigcap_n \overline{U_{x|_n}} \neq \emptyset$, by compactness. Hence, the Axiom of Choice gives an injection of $\mathcal{C}$ into X. □

From this we get the following corollary, which also follows from amenability of $\mathbb{Z}$:

Corollary 2. *There is no compact Hausdorff topology on $\mathbb{Z}$ making the translation action of $\mathbb{Z}$ on itself continuous.*

Proof. Assume for contradiction that there is such a topology τ. If there is an isolated point, then all points are isolated, by the continuity of the translation action, which contradicts compactness. Thus $(\mathbb{Z}, \tau)$ is perfect, contradicting the above proposition. □

Recall that a topological space X is called totally disconnected if every maximal connected component in X is a singleton.

Proposition 3. *Any normal T_1 topological space X of cardinality less than continuum is totally disconnected.*

Proof. Let $Y \subseteq X$ be connected and assume for contradiction that there are distinct $x, y \in Y$. Since X is T_1, the singletons $\{x\}, \{y\}$ are closed. Thus, By Urysohn's lemma, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$. But then $f(Y)$ is connected and hence must contain $[0, 1]$, contradicting Y being less than continuum. □

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Recall that X is called zero-dimensional if it admits a basis of clopen sets. Clearly, zero-dimensional implies totally disconnected, for T_1 spaces. The converse fails in general (even for metric spaces), but holds for locally compact Hausdorff spaces. The proof of this is not hard and can be found, for example, in [AT08] (Proposition 3.1.7). This and Proposition 3 together imply:

Corollary 4. *Any locally compact normal T_1 space of cardinality less than continuum is zero-dimensional. In particular, any countable compact Hausdorff space is zero-dimensional.*

Lemma 5. *Any countable compact Hausdorff space X is first-countable.*

Proof. By Corollary 4, X is zero-dimensional. Let $\mathcal{U}$ be the collection of all open sets $V \subseteq X$ such that for all $x \in V$, there is a countable neighborhood base at x. Let U be the union of all sets in $\mathcal{U}$ and put $K = U^c$.

Claim. K is perfect in the relative topology.

Proof of Claim. Assume for contradiction that there is $x \in K$ that is isolated in K. Hence there is a clopen neighborhood $V \subseteq X$ of x such that $V \cap K = \{x\}$. Note that x is not an isolated point in X as otherwise $x \in U$. This implies in particular that V is infinite (X is Hausdorff). Enumerate $V \setminus \{x\} = \{x_n\}_{n\in\mathbb{N}}$ and inductively construct a decreasing sequence $(V_n)_{n\in\mathbb{N}}$ of clopen neighborhoods of x such that

(i) $V_0 = V$;
(ii) $V_n \subseteq V$;
(iii) $x_n \notin V_{n+1}$.

Let $V_0 = V$ and assume V_n is constructed. Note that V_n is not a singleton as x is not isolated in X, and let $k \in \mathbb{N}$ be the least such that $x_k \in V_n$. Take $V_{n+1} \subseteq V_n$ to be a clopen neighborhood of x not containing x_k.

We show that the sets V_n form a neighborhood base at x. First, put $U_n = V_n \setminus V_{n+1}$ and note that for every $m \in \mathbb{N}$,

$$V_m = \{x\} \cup \bigcup_{n\geq m} U_n. \tag{$*$}$$

Now let $W \subseteq X$ be a clopen neighborhood of x. Note that $W \supseteq U_n$, for all but finitely many $n \in \mathbb{N}$, as otherwise, by (i) and $(*)$, $\{W \cap V\} \cup \{U_n \setminus W : n \in \mathbb{N}\}$ would be an infinite pairwise disjoint open cover of V, contradicting the compactness of V. By $(*)$ again, this implies that $V_m \subseteq W$, for some $m \in \mathbb{N}$.

Thus x has a countable neighborhood base and hence $V \in \mathcal{U}$, contradicting $x \in K$. ⊠

By Proposition 1, K has to be empty and hence X is first-countable. □

Corollary 4 and Lemma 5 imply the following theorem:

Theorem 6. *Every countable compact Hausdorff space X is Polish and zero-dimensional.*

Proof. We only need to note that if a countable X is first-countable, then it is second-countable. Hence by the Urysohn metrization theorem, X is metrizable and thus Polish. □

References

[AT08] A. Arhangel'skii and M. Tkachenko, *Topological Groups And Related Structures*, Atlantis Studies in Mathematics, vol. 1, Atlantic Press, 2008.