SEGAL'S EFFECTIVE WITNESS TO MEASURE-HYPERFINITENESS[1]

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Let X be the Baire space (or any other recursively presented Polish space), E a countable Δ^1_1 equivalence relation on X and μ a Borel probability measure on X.

The aim of this note is to prove the following theorem.

Theorem 1 (Segal). *If E is μ-hyperfinite, then there exists a $\Delta^1_1(\mu)$ sequence $E_0 \subseteq E_1 \subseteq \dots$ of Δ^1_1 finite equivalence relations on X and a $\Delta^1_1(\mu)$ μ-conull set $A \subseteq X$ such that $E|_A = \bigcup_n E_n|_A$.*

We start with explicitly stating the effective version of the Feldman-Moore theorem.

Proposition 2 (Feldman-Moore). *There exists a Δ^1_1 sequence of involutions such that E is equal to the union of their graphs; more precisely, there is a Δ^1_1 function $f : \mathbb{N} \times X \to X$ such that for each n, f_n is an involution, and $E = \bigcup_{n \in \mathbb{N}} \text{graph}(f_n)$.*

Proof. The usual proof of the Feldman-Moore theorem is effective provided that one uses the effective version of Lusin-Novikov (see for example 4F.17 of [Mos80]). □

Below let $(f_i)_{i \in \mathbb{N}}$ denote a Δ^1_1 sequence of involutions as above and without loss of generality assume that $f_0 = \text{id}$. For any subequivalence relation F of E, put $A_{n,F} = \{x \in X : \forall i < n (xFf_i(x))\}$ and $d_n(F, E) = 1 - \mu(A_{n,F})$ (think of $d_n(F, E)$ as the distance between F and the n^{th} approximation of E).

The content and the proof of the following lemma are very similar to those of Lemma 3.1 in [Kec12].

Lemma 3. *Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers whose sum converges, and let $(F_n)_{n \in N}$ be a sequence of Borel subequivalence relations of E such that $d_n(F_n, E) < \varepsilon_n$. Then $E = \bigcup_{n \in \mathbb{N}} E_n$ μ-a.e., where $E_n = \bigcap_{m \geq n} F_m$.*

Proof. Let $N = \limsup_n A^c_{n,F_n}$ and note that by the Borel-Cantelli lemma, N is μ-null since the sequence $\mu(A^c_{n,F_n}) < \varepsilon_n$ is summable. Now for any $x \notin N$, there is n such that for all $m \geq n$, $x \in A_{m,F_m}$. Thus for any i and $m \geq \max(i, n) =: k$, $xF_m f_i(x)$, and hence $xE_k f_i(x)$. Therefore, $[x]_E = \bigcup_{k \in N} [x]_{E_k}$. □

From this we get the following measure-theoretic characterization of hyperfiniteness that enables using approximations by finite equivalence relations instead of increasing unions (replaces qualitative with quantitative).

Proposition 4 (Conley–Miller). *E is μ-hyperfinite if and only if for all $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists a finite subequivalence relation F of E such that $d_n(F, E) < \varepsilon$.*

Proof. $\Rightarrow$: Choose F_n to be such that $d_n(F_n, E) < \frac{1}{2^n}$ and apply Lemma 3.
$\Leftarrow$: Let $(E_n)_{n \in \mathbb{N}}$ be an increasing sequence of Borel subequivalence relations with $E = \bigcup_{n \in \mathbb{N}} E_n$ μ-a.e. Given $n \in \mathbb{N}$ and $\varepsilon > 0$, note that $(A_{n,E_m})_{m \in \mathbb{N}}$ is increasing and $\bigcup_{m \in \mathbb{N}} A_{n,E_m}$ is a μ-conull set. Hence, for some m, $\mu(A_{n,E_m}) > 1 - \varepsilon$, and thus taking $F = E_m$ works. □

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[1]This note is almost entirely taken from the first section of [CM12].

Now we effectively construct such finite approximations. Let $(U_n)_{n\in\mathbb{N}}$ be a recursive basis for X that is closed under finite unions. For each $s \in \mathbb{N}^{<\omega}$, let F_s denote the smallest equivalence relation containing $\bigcup_{m<n} \mathrm{graph}(f_m|_{U_{s(m)}})$. A simple computation shows that the sequence $(F_s)_{s\in\mathbb{N}^{<\omega}}$ is Δ^1_1 (as a subset of $\mathbb{N}^{<\omega} \times X^2$).

The following lemma shows that these F_s approximate any finite subequivalence relation of E (although F_s itself may not be a finite equivalence relation).

Lemma 5 (Conley–Miller)**.** *If μ is E-quasi-invariant, then for any finite Borel subequivalence relation F of E and $\varepsilon > 0$, there is $s \in \mathbb{N}^{<\omega}$ such that $\mu(\{x \in X : [x]_{F_s} \neq [x]_F\}) < \varepsilon$.*

Proof. For each n, put $X_n = \{x \in X : [x]_F \subseteq \{f_m(x) : m < n\}\}$. Clearly X_n are increasing and, because F is finite, $X = \bigcup_{n\in\mathbb{N}} X_n$. Thus there is $n \in \mathbb{N}$ such that $\mu(X_n) > 1 - \frac{\varepsilon}{2}$. Now for each $m < n$, put $A_m = \{x \in X_n : xFf_m(x)\}$ and note that the smallest equivalence relation F' containing $\bigcup_{m<n} \mathrm{graph}(f_m|_{A_m})$ has the property that for all $x \in X_n$, $[x]_H = [x]_F$. Thus, to prove the lemma, we only need to replace the sets A_m by some basic open sets from $(U_n)_{n\in\mathbb{N}}$, which we do as follows: put $f_{ij} = f_i \circ f_j$, for $i, j < n$, and use the regularity and E-quasi-invariance of μ to get $k_m \in \mathbb{N}$ such that for all $i, j < n$,

$$\mu(f_{ij}^{-1}(U_{k_m} \Delta A_m)) < \frac{\varepsilon}{2n^3}.$$

Define $s \in \mathbb{N}^n$ by $s(m) = k_m$, for $m < n$, and put

$$\begin{aligned} Y &:= X_n \setminus \left(\bigcup_{i,j,m<n} f_{ij}^{-1}(U_{s(m)} \Delta A_m) \right) \\ &= \left\{ x \in X_n : \forall i, j, m < n (f_{ij}(x) \in U_{s(m)} \leftrightarrow f_{ij}(x) \in A_m) \right\}, \end{aligned}$$

noting that $\mu(Y) > 1 - \varepsilon$, so it remains to show that for any $x \in Y$, $[x]_{F_s} = [x]_F$.

We chose the sets $U_{s(m)}$ and defined Y like this to ensure that for all $x \in Y$, all of the points somehow related to x that we consider in the proof below are in

$$Z := \left\{ x \in X_n : \forall m < n (x \in U_{s(m)} \iff x \in A_m) \right\}.$$

More precisely, fix $x \in Y$ and note that

(i) $\{f_{ij}(x) : i, j < n\} \subseteq Z$,
(ii) $\{f_m(x) : m < n\} \subseteq Z$,
(iii) $[x]_F \subseteq Z$,

where (i) implies (ii) because f_0 is the identity, and (ii) implies (iii) because $x \in X_n$. Now we show that $[x]_{F_s} = [x]_F$.

$\boldsymbol{[x]_{F_s} \supseteq [x]_F}$**:** Fix $y \in [x]_F$. Since $x \in X_n$, there is $m < n$ such that $y = f_m(x)$ and thus $x \in A_m$. But because $x \in [x]_F \subseteq Z$, it follows that x is also in $U_{s(m)}$. Hence $(x, y) \in \mathrm{graph}(f_m|_{U_{s(m)}})$, so xF_sy.

$\boldsymbol{[x]_{F_s} \subseteq [x]_F}$**:** Take $y \in [x]_{F_s}$. Thus, by the definition of F_s, there exist $x_0, ..., x_l \in X$ such that $x_0 = x, x_l = y$ and for each $k < l$, there is $i < n$ such that $x_{k+1} = f_i(x_k)$ and at least one of x_k, x_{k+1} belongs to $U_{s(i)}$. We show by induction on k that xFx_k. Suppose xFx_k and show that x_kFx_{k+1}. Let $i < n$ be such that $x_{k+1} = f_i(x_k)$ and at least one of x_k, x_{k+1} is in $U_{s(i)}$. Also let $j < n$ be such that $f_j(x) = x_k$ (such j exists since $x \in X_n$). Now note that $x_k \in [x]_F \subseteq Z$ by (iii), and $x_{k+1} = f_{ij}(x) \in Z$ by (i). Thus, no matter which one of x_k, x_{k+1} is in $U_{s(i)}$, it would also be in A_i, and therefore x_kFx_{k+1} (here we use that f_i is an involution). □

The following lemma is used to define a finite version of F_s.

Lemma 6. *The set $B = \{(s,x) \in \mathbb{N}^{<\omega} \times X : [x]_{F_s}$ is finite$\}$ is Δ^1_1.*

Proof. For any $(s,x) \in \mathbb{N}^{<\omega} \times X$,

$$(s,x) \in B \iff \exists n \in \mathbb{N} \forall m \in \mathbb{N}(x F_s f_m(x) \to f_m(x) \in \{f_i(x) : i < n\}).$$

Clearly, the right hand side of this is a Δ^1_1 definition. □

Now define F'_s to be F_s on B_s and the identity relation elsewhere. It is clear that in Lemma 5, we can replace F_s with F'_s. This and Proposition 4 imply the following:

Corollary 7. *Suppose μ is E-quasi-invariant. Then E is μ-hyperfinite if and only if for all $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $s \in \mathbb{N}^{<\omega}$ such that $d_n(F'_s, E) = 1 - \mu(A_{n,F'_s}) < \varepsilon$.*

The following lemma will imply that the right side of Corollary 7 is $\Delta^1_1(\mu)$.

Lemma 8. *If $C \subseteq \mathbb{N} \times X$ is Δ^1_1, then the set $D \subseteq \mathbb{N} \times \mathbb{Q}$ defined by*

$$(n,q) \in D \iff \mu(C_n) > q$$

is $\Delta^1_1(\mu)$.

Proof. By applying the same to the complement of C, it is enough to show that D is $\Sigma^1_1(\mu)$. This can be done by applying basically the same proof as that of Theorem 2.2.3 of [Kec73]. □

Now we are ready to prove the main theorem.

Proof of Theorem 1. Firstly, note that without loss of generality we can assume that μ is quasi-invariant since we can always replace it with $\mu' = \sum_{n \geq 1} 2^{-n} f^*_n \mu$ as μ' and μ agree on E-invariant sets and hence it will not affect the statement of the theorem.

Define $C \subseteq \mathbb{N}^{<\omega} \times \mathbb{N} \times X$ by

$$(s,n,x) \in C \iff \forall m < n(x F'_s f_m(x)).$$

Applying Lemma 8 to C, the function $(s,n) \mapsto \mu(A_{n,F'_s})$ is $\Delta^1_1(\mu)$. Thus so is the function $\pi : \mathbb{N} \to \mathbb{N}^{<\omega}$ defined by $n \mapsto$ the least $s \in \mathbb{N}^{<\omega}$ such that $1 - \mu(A_{n,F'_s}) < 2^{-n}$ (such s exists by Corollary 7). Now set $E_n = \bigcap_{m \geq n} F'_{\pi(m)}$ and put $E' = \bigcup_{n \in \mathbb{N}} E_n$. Finally, let $Z = \{x \in X : [x]_E \neq [x]_{E'}\}$ and note that Z is $\Delta^1_1(\mu)$ since $x \in Z \iff \exists n \in \mathbb{N} \neg(x E' f_n(x))$. Moreover, by Lemma 3, Z is μ-null, so we are done. □

References

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