HJORTH'S PROOF OF THE EMBEDDABILITY OF HYPERFINITE EQUIVALENCE RELATIONS INTO E_0

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The following theorem was proven in [DJK94, Theorem 7.1]. This note gives an alternative proof by Greg Hjorth.

Theorem (Dougherty–Jackson–Kechris)**.** *Every hyperfinite equivalence relation is Borel embeddable into E_0.*

Proof (Hjorth). Without loss of generality assume that $X = 2^{\mathbb{N}}$ and E is a hyperfinite equivalence relation on X with $E = \bigcup_{n \in \mathbb{N}} F_n$, where $\{F_n\}$ is an increasing sequence of finite Borel equivalence relations on X with F_0 being the identity relation. Let $\mathcal{N}$ denote the Baire space. It is enough to show that E is Borel embeddable into E_0 on $\mathcal{N}$ (eventual agreement of sequences of natural numbers).

Fix a Borel linear ordering $<$ on X. Define a map $f : X \to \mathcal{N}$ by $x \mapsto \sigma_x$, where each $\sigma_x(n)$ is defined as follows: if $[x]_{F_n} = \{y_0^n, ..., y_{k_n}^n\}$ with $y_0^n < ... < y_{k_n}^n$ then

- $\sigma_x(0)$ is the code (a natural number) of $(y_0^0\restriction_0; 0)$ (in some a priori fixed coding);
- for $n \geq 1$, $\sigma_x(n)$ is the code of

$$(y_0^n\restriction_n, y_1^n\restriction_n, ..., y_{k_n}^n\restriction_n; i(n,0), i(n,1), ..., i(n,k_{n-1})),$$

where $i(n,j) \in \mathbb{N}$ and $y_j^{n-1} = y_{i(n,j)}^n$ for all $j \leq k_{n-1}$.

It is clear from the definition that f is a Borel homomorphism from E to E_0 since $\forall x, y \in X$, if xEy then xF_ny for some n and hence $\sigma_x(m) = \sigma_y(m)$, for all $m \geq n+1$.

Claim. For all $x, y \in X$ and $n \in \mathbb{N}$, if $\forall m \geq n, \sigma_x(m) = \sigma_y(m)$ then xF_ny.

Proof of Claim. Let $[x]_{F_n} = \{y_0^n, ..., y_{k_n}^n\}$ with $y_0^n < ... < y_{k_n}^n$. It is enough to show that we can recover every y_j^n from $\sigma_x\restriction_{[n,+\infty)}$, for $j \leq k_n$. Indeed, for all $m \geq n$, let $\sigma_x(m)$ be equal to the code of

$$(s_0^m, s_1^m, ..., s_{k_m}^m; i(m,0), i(m,1), ..., i(m,k_{m-1})),$$

where $s_j^m \in \mathbb{N}^m$ and $i(m,l) \in \mathbb{N}$, for all $j \leq k_m, l \leq k_{m-1}$. Fix $j \leq k_n$ and define $J : \{n, n+1, ...\} \to \mathbb{N}$ recursively by

$$J(m) = \begin{cases} j & \text{if } m = n \\ i(m, J(m-1)) & \text{if } m \geq n+1 \end{cases}.$$

By the definitions of σ_x and J, $s_{J(m)}^m \sqsubseteq s_{J(m+1)}^{m+1}$, for all $m \geq n$, and $y_j^n = \bigcup_{m \geq n} s_{J(m)}^m$. ⊠

This claim implies that f is a Borel reduction from E to E_0. Moreover, it implies that f is an embedding since F_0 is the identity relation on X. □

References

[DJK94] R. Dougherty, S. Jackson, and A. S. Kechris, *The Structure of Hyperfinite Borel Equivalence Relations*, Trans. of the Amer. Math. Soc. **341** (1994), no. 1, 193–225.

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