REALIZATIONS OF GRAPHS IN A HILBERT SPACE

TERENCE TAO AND ANUSH TSERUNYAN

The purpose of this note is to show that any countable graph admits a realization as an orthogonality graph in a Hilbert space, while providing an uncountable counter-example.

Proposition 1. *Every countable graph $\mathcal{G} = (V, E)$ is realizable as an orthogonality graph in a Hilbert space; more precisely, to every vertex in V, one can assign a vector in a separable Hilbert space such that for any $u, v \in V$, $u \perp v$ if and only if uEv.*

Proof. Writing $V = (v_n)_{n\in\mathbb{N}}$ and the vector 0 to all the v_n. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in a separable Hilbert space. The construction will be done by induction on n, where at the n^{th} stage, the only nonzero vectors will be among $v_1, v_2, ..., v_n$ with norms at most $\sum_{k=1}^{n} 2^{-k}$ and such that $v_k \in \operatorname{Span}(e_k, ..., e_n)$ for each $k \leq n$; moreover, the restricting of $\mathcal{G}$ to these vertices coincides with their orthogonality graph. Now put $v_1 \coloneqq e_1$ and suppose that $v_1, v_2, ..., v_n$ are as desired and put $v_n \coloneqq 2^{-(n+1)} e_{n+1}$. Let $D \subseteq \{v_k\}_{k \leq n}$ be the set of all vertices nonadjacent to v_n in $\mathcal{G}$. Redefine each $v \in D$ as follows: $v \coloneqq v + 2^{-(n+1)} e_{n+1}$. □

We now work towards the counter-example in the uncountable case.

Observation 2. *Let $A = \{u_i\}_{i=1}^{n}$, $n \geq 2$, be a set of unit vectors in complex vector space such that any two distinct vectors in A form the same angle $\theta \in \mathbb{C}$; more precisely, for all distinct $i, j \leq n$, $\langle u_i, u_j \rangle = \theta$. Then $\theta \in \mathbb{R}$ and $\theta \geq 1/(n-1)$.*

Proof. First off, since $\langle u_1, u_2 \rangle = \theta = \langle u_2, u_1 \rangle$, θ must be real. Now consider $u = u_1 + u_2 + ... + u_n$ and write that its norm is non-negative:

$$\langle u, u \rangle = n + n(n-1)\theta \geq 0.$$

□

The following lemma is proved by a cool averaging trick that comes from the general ergodic theorem that involves the closure of the convex hull.

Lemma 3 (Tao). *Let I be an infinite set and let $A = (u_i)_{i\in I}$ be a sequence of unit vectors in a Hilbert space $\mathcal{H}$ such that for any two distinct $i, j \in I$, the corresponding vectors form the same angle; more precisely, there is $\theta \in \mathbb{C}$ such that $\forall i \neq j \in I$, $\langle u_i, u_j \rangle = \theta$. Then θ is non-negative real and these vectors have "an average", i.e. there is a vector v of norm $\sqrt{\theta}$ in the closed convex hull of A such that, for every $i \in I$, $u_i = v + \sqrt{1-\theta}\, w_i$ with $w_i \perp v$ and such that $\{w_i\}_{i\in I}$ is orthonormal.*

Proof. The non-negativity of θ follows from the above observation. For the rest, let $(u_n)_{n\in\mathbb{N}}$ be a countable sequence from A and, for each $n \in \mathbb{N}$, put

$$v_n \coloneqq \frac{1}{n}(u_1 + u_2 + ... + u_n).$$

Date: July, 2015.

Claim. The sequence $(v_n)_{n\in\mathbb{N}}$ is Cauchy.

Proof. For $n < m \in \mathbb{N}$, we compute:

$$\begin{aligned}
\|v_m - v_n\|^2 &= \langle v_m - v_n, v_m - v_n\rangle \\
&= \langle v_n, v_n\rangle + \langle v_m, v_m\rangle - 2\operatorname{Re}\langle v_n, v_m\rangle \\
&= \frac{1}{n^2}[n + n(n-1)\theta] + \frac{1}{m^2}[m + m(m-1)\theta] - \frac{2}{nm}[n + n(m-1)\theta] \\
&= \frac{1}{n}[1 + (n-1)\theta] + \frac{1}{m}[1 + (m-1)\theta] - \frac{2}{m}[1 + (m-1)\theta] \\
&= \frac{1}{n}[1 + (n-1)\theta] - \frac{1}{m}[1 + (m-1)\theta] \\
&\to 0 \text{ as } n, m \to \infty.
\end{aligned}$$

□

Note that $\langle v_n, v_n\rangle = \frac{1}{n^2}[n + n(n-1)\theta] \to \theta$, so $\|v\| = \sqrt{\theta}$. Moreover, since for any $i \in I$ and $n \in \mathbb{N}$, $\langle u_i, v_n\rangle$ is equal to either $\frac{n\theta}{n} = \theta$ or $\frac{1+(n-1)\theta}{n}$, $\langle u_i, v\rangle = \theta$.

For each $i \in I$, write the orthogonal decomposition of u_i over v: $u_i = v + \sqrt{1-\theta}w_i$, so w_i is a normal vector orthogonal to v. Now for any distinct $i, j \in I$, $\theta = \langle u_i, u_j\rangle = \langle v, v\rangle + \langle w_i, w_j\rangle = \theta + \langle w_i, w_j\rangle$, so $\langle w_i, w_j\rangle = 0$. □

We now give an example of an uncountable (in fact, bipartite) graph that doesn't admit an orthogonality graph realization in a Hilbert space.

Example 4. Let I be a set of cardinality $\left(2^{(2^{\mathfrak{c}})}\right)^+$ and let $U \coloneqq \{u_J\}_{J\in\mathscr{P}(I)}$, $V \coloneqq \{v_i\}_{i\in I}$ be our disjoint partitions of vertices. Put an edge between u_J and v_i if and only if $i \in J$.

Proposition 5. *The graph in Example 4 doesn't have an orthogonality graph realization in a Hilbert space.*

Proof. Suppose it does, so assume that u_J, v_i are vectors in some Hilbert space $\mathcal{H}$ such that $i \in J$ if and only if $u_J \perp v_i$. Since there are only continuum-many possibilities for $\langle v_i, v_j\rangle$, $i, j \in I$, Erdős–Rado gives an uncountable (in fact, $(2^{\mathfrak{c}})^+$) set $J \subseteq I$ and a nonzero $\theta \in \mathbb{C}$ such that for any distinct $i, j \in J$, $\langle v_i, v_j\rangle = \theta$. By Lemma 3, $\theta > 0$ and there is a vector $v \in \mathcal{H}$ of norm $\sqrt{\theta}$ such that for every $i \in J$, $v_i = v + \sqrt{1-\theta}w_i$, where $w_i \perp v$ and the sequence $(w_i)_{i\in J}$ is orthonormal.

Now partition J into two uncountable sets J_0 and J_1. Note that u_{J_0} is orthogonal to every v_i with $i \in J_0$, so it must be orthogonal to v as well since otherwise, it would have to be nonorthogonal to every w_i with $i \in J_0$, contradicting Bessel's inequality. On the other hand, u_{J_0} is nonorthogonal to every v_j with $j \in J_1$, so, since it's orthogonal to v, it must be nonorthogonal to every w_j with $j \in J_1$, again contradicting Bessel's inequality. □