THE EFFROS SPACE OF A σ-POLISH SPACE IS STANDARD BOREL

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Call a topological space X *σ-Polish* if it admits a countable basis $(U_n)_{n\in\mathbb{N}}$ such that each U_n is Polish in the relative topology. Such topologies arise naturally when considering Polish groupoids and this note gives a positive answer to a question of Martino Lupini as to whether the Effros structure of such topological spaces is standard Borel.

Let $F(X)$ denote the collection of the closed subsets of X. We endow $F(X)$ with the σ-algebra $\mathcal{E}$ generated by the sets of the form

$$\mathcal{F}_U := \{F \in F(X) : F \cap U \neq \emptyset\},$$

for $U \subseteq X$ open, and call the measurable space $(F(X), \mathcal{E})$ the Effros space of X. It is clear that $\mathcal{E}$ is generated by the sets $\mathcal{F}_{U_n}$.

Theorem. *The Effros space of X is standard. In fact, the topology generated by the sets $\mathcal{F}_{U_n}$ and $\mathcal{F}^c_{U_n}$ is Polish.*

Proof. Consider the coding map $c : F(X) \to 2^{\mathbb{N}}$ given by $F \mapsto$ the characteristic function of $\{n \in \mathbb{N} : F \cap U_n \neq \emptyset\}$. It is clear that c is measurable with respect to $\mathcal{E}$ since the preimage of a prebasic open set $\{y \in 2^{\mathbb{N}} : y(n) = i\}$, $i \in \{0, 1\}$, is $\mathcal{F}_{U_n}$ or its complement, depending on whether $i = 1$ or 0. Conversely, c maps the sets $\mathcal{F}_{U_n}$ to basic open sets $\{y \in 2^{\mathbb{N}} : y(n) = 1\}$, so $c^{-1} : Y \to F(X)$ is also measurable, where $Y = c(F(X))$. This makes c an isomorphism between measurable spaces $(F(X), \mathcal{E})$ and $(Y, \mathcal{B}(2^{\mathbb{N}})|_Y)$. But if Y is Borel, the latter measurable space is standard Borel and hence so would be the former.

We show in fact that Y is a G_δ subset of $2^{\mathbb{N}}$. Indeed, fixing a compatible complete metric d_n on U_n, we have the following claim, which immediately implies that Y is G_δ.

Claim. For each $y \in 2^{\mathbb{N}}$, $y \in Y$ if and only if the following conditions hold:

(i) for all n, m with $U_n \subseteq U_m$, if $y(n) = 1$ then $y(m) = 1$;

(ii) for all n and $\varepsilon \in \mathbb{Q}^+$, if $y(n) = 1$ then there is m such that $y(m) = 1$ and for all $i \leq n$ with $U_i \supseteq U_n$, we have:

$$\overline{U_m}^i \subseteq U_n \text{ and } \operatorname{diam}_i(U_m) < \varepsilon,$$

where the closure $\overline{U_m}^i$ and diameter $\operatorname{diam}_i(U_m)$ are taken with respect to the metric d_i.

Proof of Claim. The left-to-right direction is straightforward, so we check the other direction. Let $y \in 2^{\mathbb{N}}$ satisfy conditions (i) and (ii), and put $F = \{x \in X : \forall n (x \in U_n \Rightarrow y(n) = 1)\}$. By definition, F is closed in X (the complement is open), and we show that $c(F) = y$. Fix n and note that if $y(n) = 0$, then $F \cap U_n = \emptyset$ by definition. So suppose $y(n) = 1$ and we have to find an $x \in F \cap U_n$. Iterating (ii), we get a sequence $(U_{n_k})_k$ with $n_0 = n$ and such that for all $k \geq 1$,

- $y(n_k) = 1$,
- $\overline{U_{n_k}}^n \subseteq U_{n_{k-1}}$,
- $\operatorname{diam}_n(U_{n_k}) \leq 1/k$.

Thus, since the metric d_n on U_n is complete, we get $\{x\} = \bigcap_k \overline{U_{n_k}}^n$, for some $x \in U_n$. It remains to show that $x \in F$, but this easily follows from (i). ⊠

Thus, inheriting the topology of Y, $F(X)$ becomes Polish, so it remains to note that the topology of Y is generated by the sets $\{y \in 2^{\mathbb{N}} : y(n) = i\}$, $i \in \{0, 1\}$, whose preimages under c are the sets $\mathcal{F}_{U_n}, \mathcal{F}^c_{U_n}$. □

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