ABSTRACT. We prove a pointwise ergodic theorem for quasi-probability-measure-preserving (quasi-pmp) locally countable measurable graphs, analogous to pointwise ergodic theorems for group actions, replacing the group with a Schreier graph of the action. For any quasi-pmp graph, the theorem gives an increasing sequence of Borel subgraphs with finite connected components along which the averages of $L^1$ functions converge to their expectations. Equivalently, it states that any (not necessarily pmp) locally countable Borel graph on a standard probability space contains an ergodic hyperfinite subgraph.

The pmp version of this theorem was first proven by R. Tucker-Drob using probabilistic methods. Our proof is different: it is descriptive set theoretic and applies more generally to quasi-pmp graphs. Among other things, it involves introducing a graph invariant, a method of producing finite equivalence subrelations with large domain, and a simple method of exploiting nonamenability of a measured graph. The non-pmp setting additionally requires a new gadget for analyzing the interplay between the underlying cocycle and the graph.

Contents

1. Introduction .................................................. 2
   Pointwise ergodic theorems for group actions .................. 2
   Quasi-pmp graphs and the main result ......................... 3
   History of the main theorem and applications ............... 6
   Our proof .................................................. 7
   Other results ............................................... 7
   Acknowledgments ........................................... 9
   Organization ............................................... 9

2. Sketch of proof ................................................ 9
   2.A. The set of $G$-asymptotic averages ....................... 10
   2.B. Packed prepartitions and finitizing cuts .................. 11
   2.C. Iteration via measure-compactness ....................... 12
   2.D. The general quasi-invariant setting ..................... 13

3. Preliminaries .................................................. 14
   3.A. Equivalence relations ................................... 14
   3.B. Cocycles ............................................... 14
   3.C. Graphs ................................................ 16

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1. Introduction

Pointwise ergodic theorems for group actions. Dating back to G.D. Birkhoff [Bir31], the pointwise ergodic theorem for a probability measure preserving⁴ (pmp) action of \( \mathbb{Z} \) on a standard probability space \((X, \mu)\) is a bridge between the global condition of the ergodicity

An action \( \Gamma \actson (X, \mu) \) of a countable group on a standard probability space is probability measure preserving if each \( \gamma \in \Gamma \) acts as a measurable automorphism of \( X \) and \( \mu(\gamma \cdot A) = \mu(A) \) for each measurable set \( A \subseteq X \).
of the action and the a.e.-local combinatorics of the induced Schreier graph of the action as an $f$-valued graph for each $f \in L^1(X,\mu)$. The locality windows for testing this are taken from the group, e.g., the intervals $[-n,n] \subseteq \mathbb{Z}$, and hence they are uniform throughout the action space and are used at all $x \in X$ at once. This was later generalized to amenable groups by E. Lindenstrauss [Lin01] with the sequence of sets $[-n,n]$ replaced by any tempered Følner sequence. It is worth pointing out that Lindenstrauss’ result but for $L^2$-functions, was proven earlier in [Shu80]; for more on ergodic theorems for amenable groups, we refer to [Wei03], [Nev06], [AAB+10].

Various versions of pointwise ergodic theorems for pmp actions have been proven for free groups by R. Grigorchuk [Gri87, Gri99, Gri00], A. Nevo [Nev94], A. Nevo and E. Stein [NS94], A. Bufetov [Buf02], and L. Bowen and A. Nevo [BN13a], [BN15, Theorems 6.2 and 6.3]; we refer to [BK12] for a survey. Furthermore, L. Bowen and A. Nevo have pointwise ergodic theorems for certain pmp actions of other groups, see [BN13b], [BN15], [BN15] and references therein.

As for quasi-pmp\footnote{An action $\Gamma \curvearrowleft (X,\mu)$ of a countable group on a standard probability space is quasi-pmp (or nonsingular) if each $\gamma \in \Gamma$ acts as a measurable automorphism of $X$ and maps null sets to null sets.} (i.e., nonsingular) actions, very little is known. A quasi-pmp pointwise ergodic theorem for $\mathbb{Z}^d$ was first proven by J. Feldman [Fel07] and then generalized in two different directions by M. Hochman [Hoc10] and by A. Dooley and K. Jarrett [DJ16]. For general groups of polynomial growth, M. Hochman obtained [Hoc13, Theorem 1.4] a slightly weaker form of a quasi-pmp ergodic theorem where the a.e. convergence is replaced with a.e. convergence in density. Several quasi-pmp pointwise ergodic theorems for lattice actions on homogeneous spaces were proven in a number of works by various authors, comprehensive references to which are given in [BN14, Subsection 1.3]. Furthermore, K. Jarrett showed [Jar17] that the quasi-pmp pointwise ergodic theorem holds for any Heisenberg group along some special sequences of finite sets, which only depends on the group and not the action. Finally, an indirect version of the quasi-pmp pointwise ergodic theorem for free groups is obtained in [BN14].

On the negative side, M. Hochman proved [Hoc13, Theorem 1.1] that the quasi-pmp pointwise ergodic theorem holds only along sequences of subsets of the group satisfying the Besicovitch covering property. He then infers [Hoc13, Theorems 1.2 and 1.3] that the quasi-pmp pointwise ergodic theorem fails for any sequence of subsets of $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}$ and any subsequence of balls in nonabelian free groups as well as in the Heisenberg group. Given these negative results, it makes sense to seek for a weaker, yet more general, version of the quasi-pmp pointwise ergodic theorem, namely: instead of deterministically taking a sequence of subsets of the group, choose them at random for every point in the action space. We make this precise in the following subsection.

Quasi-pmp graphs and the main result

As mentioned above, a (quasi-)pmp action $\alpha$ of a countable group $\Gamma$ on $(X,\mu)$ induces a (quasi-)pmp (see Definitions 3.1 and 3.7) locally countable graph $G_S$ on $X$ once we fix a generating set $S$ for $\Gamma$:

$$xG_S y \iff \sigma \cdot x = y \text{ for some } \sigma \in S.$$  

We refer to this as the Schreier graph of the action $\alpha$ (with the generating set $S$). The pointwise ergodic theorems extract a sequence $(F_n)$ of subsets of (or, more generally, measures on)
Theorem 1.1 (Pointwise ergodic for quasi-pmp graphs). Let $G$ be a locally countable quasi-pmp ergodic Borel graph on a standard probability space $(X, \mu)$ and let $\rho : E_G \to \mathbb{R}^+$ be the Radon–Nikodym cocycle corresponding to $\mu$. There is an increasing sequence $(F_n)$ of $G$-connected finite Borel equivalence relations such that for any $f \in L^1(X, \mu)$,

$$\lim_{n \to \infty} A^\rho_f[F_n](x) = \int f \, d\mu \text{ for a.e. } x \in X.$$ 

We briefly explain some of the terminology here, referring to Section 3 for the rest.

- $E_G$ denotes the connectedness equivalence relation of a graph $G$.
- The terms quasi-pmp and ergodic applied to $G$ mean the same for $E_G$.
- We say that the cocycle $\rho : E_G \to \mathbb{R}^+$ corresponds to $\mu$ if $\rho$ is the (unique) Radon–Nikodym cocycle making $\mu \rho$-invariant, see [KM04, Section 8].
- An equivalence relation $F$ is called finite (resp. countable) if each $F$-class is finite (resp. countable).
- Call an equivalence relation $F$ on $X$ $G$-connected if each $F$-class is $G$-connected; equivalently, $G \cap F$ is a graphing of $F$.
- For $F := F_n$, the function $x \mapsto A^\rho_f[F](x)$ is defined by mapping $x \in X$ to the $\rho$-weighted average of $f$ over the $F$-equivalence class $[x]_F$ of $x$, i.e.

$$A^\rho_f[F](x) := \frac{\sum_{y \in [x]_F} f(y) \rho(y, x)}{\sum_{y \in [x]_F} \rho(y, x)}.$$ 

Remark 1.2. In Theorem 1.1, the condition of $G$-connectedness on $F_n$ is written in bold because it is the main content and difficulty of the theorem. Without it, the theorem has been known for some time now and is not hard to prove. Indeed, the pmp version is explicitly stated and proven in [Kec10, Theorem 3.5], but, even for the quasi-pmp setting, this is easily extracted from earlier works, namely, by putting together [Sch77, Theorem 8.22] and the Hurewicz ergodic theorem. Indeed, if we do not demand each $F_n$-class to be $G$-connected, but just be within one $G$-connected component, then a relatively simple exhaustion argument for building the $F_n$ would work. However, fulfilling the $G$-connectedness condition amounts to uniformly tiling the $G$-connected components with finite $G$-connected sets with roughly correct $f$-averages and this is rather challenging: if points $x, y$ need to be included in one tile to get the $f$-average right, one has to also include a path connecting them, which may destroy the desired $f$-average.

\footnote{We say that a set $U \subseteq X$ is $G$-connected if $G \cap U^2$ is a connected graph, i.e. any two points in $U$ are connected by a $G$-path that entirely lies in $U$.}
Remark 1.3. The natural version of Theorem 1.1 for nonergodic graphs also holds replacing the expectation of $f$ with its conditional expectation on the $\sigma$-algebra of all $E_G$-invariant Borel sets. We only give the proof for ergodic graphs to keep the matters simple, but our proof is Borel-uniform over $X$, so it also yields the nonergodic version.

Theorem 1.1 is equivalent to the following simpler statement.

**Theorem 1.4 (Ergodic hyperfinite subgraph).** Every locally countable ergodic Borel graph $G$ on a standard probability space $(X, \mu)$ admits an ergodic hyperfinite Borel subgraph $H \subseteq G$.

Here, $H$ being hyperfinite means that it is an increasing union of component-finite\(^4\) Borel subgraphs (equivalently, $E_H$ is a hyperfinite equivalence relation). Note that the quasi-pmp requirement is omitted because we can always reduce to it by the standard argument of moving the measure around $E_G$.

**Remark 1.5.** As mentioned in Remark 1.2, the main difficulty of Theorem 1.4 is obtaining a subgraph of $G$ and not just a subequivalence relation of $E_G$.

The equivalence of Theorems 1.1 and 1.4 hinges on a pointwise ergodic theorem for quasi-pmp hyperfinite equivalence relations (equivalently, amenable equivalence relations, due to the Connes–Feldman–Weiss theorem [CFW81]), whose various versions have appeared in the literature, for example, in [BN15] and in [MT17, Theorem 7.3], and which is not very hard to prove. Here we state the ergodic quasi-pmp version of [MT17, Theorem 7.3], see Theorem 4.4 below for a nonergodic version.

**Theorem 1.6 (Pointwise ergodic for hyperfinite equivalence relations).** Let $F$ be a quasi-pmp ergodic hyperfinite equivalence relation on $(X, \mu)$ and let $\rho : F \to \mathbb{R}^+$ be a Borel cocycle corresponding to $\mu$. For any witness\(^5\) $(F_n)$ to the hyperfiniteness of $F$ and any $f \in L^1(X, \mu)$,

$$\lim_{n \to \infty} A^\rho_f [F_n](x) = \int_X f \, d\mu \text{ for a.e. } x \in X.$$

Finally, the fact that Theorem 1.1 is true for quasi-pmp graphs, not just pmp, implies a seemingly more general ratio ergodic theorem.

**Theorem 1.7 (Pointwise ratio ergodic theorem for quasi-pmp graphs).** Let $G$ be a locally countable quasi-pmp ergodic Borel graph on a standard probability space $(X, \mu)$, let $\rho : E_G \to \mathbb{R}^+$ be the Radon–Nikodym cocycle corresponding to $\mu$. There is an increasing sequence $(F_n)$ of $G$-connected finite Borel equivalence relations such that for any $f, g \in L^1(X, \mu)$ with $g$ positive,

$$\lim_{n \to \infty} \frac{\sum_{y \in [x]_{F_n}} f(y) \rho(y, x)}{\sum_{y \in [x]_{F_n}} g(y) \rho(y, x)} = \frac{\int_X f \, d\mu}{\int_X g \, d\mu} \text{ for a.e. } x \in X.$$

**Remark 1.8.** As mentioned above, it was shown in [Hoc13] that the deterministic analogues of Theorem 1.7 fail for quasi-pmp actions of along all or some deterministic increasing sequences of subsets of certain groups, so Theorem 1.7 is sharp in the sense that even though deterministic sequences do not work, random ones do.

\(^4\)That is: each connected component is finite.

\(^5\)We call $(F_n)$ a witness to the hyperfiniteness of $F$ if it is an increasing sequence of finite Borel subequivalence relations whose union is $F$. 

5
**History of the main theorem and applications.** Theorem 1.4 in the case of pmp graphs was first announced in 2016 by R. Tucker-Drob, who shared a sketch of his proof with the present author in personal communication. To the best of the author’s knowledge, it is still unpublished, so we state Tucker-Drob’s theorem here for reference:

**Theorem 1.9 (Tucker-Drob, 2016).** Every locally countable pmp ergodic Borel graph $G$ on a standard probability space $(X, \mu)$ admits an ergodic hyperfinite Borel subgraph $H \subseteq G$.

Even for pmp graphs, our proof is fundamentally different from Tucker-Drob’s, but before elaborating on this further, we mention a couple of applications of Theorem 1.9, and more generally, of Theorem 1.4.

An immediate corollary of Theorem 1.9 is a positive answer to a question of L. Bowen as to whether every pmp ergodic countable treeable equivalence relation has an ergodic hyperfinite free factor\(^6\).

**Corollary 1.10 (Tucker-Drob, 2016).** Every pmp ergodic countable treeable Borel equivalence relation admits an ergodic hyperfinite free factor.

Applying Theorem 1.4 instead of Theorem 1.9 removes “pmp” from the hypothesis:

**Corollary 1.11.** Every ergodic countable treeable Borel equivalence relation admits an ergodic hyperfinite free factor.

A weaker form of Theorem 1.9 is also applied in [MT17, Theorems 1.3 and 1.4] in the proof of the following strengthening of G. Hjorth’s lemma for cost attained [Hjo06]:

**Theorem 1.12 (Miller–Tserunyan, 2017).** If a countable pmp ergodic Borel equivalence relation $E$ is treeable and has cost $n \in \mathbb{N} \cup \{\infty\}$, then it is induced by an a.e. free pmp action of $F_n$ such that each of the $n$ standard generators of $F_n$ alone acts ergodically.

Tucker-Drob’s proof of Theorem 1.9 is based on a deep recent result in probability theory by T. Hutchcroft and A. Nachmias on indistinguishability of the Wired Uniform Spanning Forest (WUFS) [HN16, Theorem 1.1]. Furthermore, the derivation of Theorem 1.9 from [HN16, Theorem 1.1] also makes use of probabilistic techniques such as Wilson’s algorithm rooted at infinity as in [GL09, Proposition 9], as well as an analogue for graphs of the Abért–Weiss theorem [AW13, Theorem 1] essentially due to H. Hatami, L. Lovász, and B. Szegedy (derived by Tucker-Drob from [HLS14, Lemmas 7.9 and 7.10]).

Aiming to make their proof of Theorem 1.12 self-contained, B.D. Miller and the present author found a more constructive and descriptive set theoretic argument to prove a weaker version of Theorem 1.9, which, however, sufficed for their application in the proof of Theorem 1.12. Referring the reader to [MT17, Definition 3.19] for the definition of a well-iterated edge slide of a graph, the statement of this weaker version is as follows.

**Theorem 1.13.** For any locally countable pmp ergodic Borel graph $G$, there is a well-iterated edge slide $G'$ of $G$ that admits an ergodic hyperfinite subgraph $H \subseteq G'$.

The proof of Theorem 1.13 involved novel tools such as asymptotic averages along a graph [MT17, Section 8], saturated and packed prepartitions/fsr [MT17, Subsections 4.C and 4.D], and finitizing edge-cuts [MT17, Section 9]. However, the argument was tailored

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\(^6\)For Borel equivalence relations $E \subseteq F$, $E$ is said to be a free factor of $F$ if $F = E \ast F'$, for some Borel equivalence relation $F'$, where $\ast$ denotes the free join in the sense of [KM04, Section 27].
to edge slides and the adjustment of the graph $G$ provided by edge sliding cannot be omitted. Furthermore, the present author does not see how to adapt that argument to the quasi-pmp setting.

The current paper advances the aforementioned tools further and, using an unusual iteration technique, provides a constructive and self-contained proof of Theorem 1.9. Moreover, some new considerations and results for cocycles on $E_G$ help adapt the proof to the quasi-invariant setting, yielding the generalization of Theorem 1.9 to all graphs on probability spaces, namely, Theorem 1.4.

**Our proof.** Tucker-Drob’s proof of Theorem 1.9 does not seem adaptable to the level of generality of Theorem 1.4 because the probabilistic statements it uses are inherently measure-preserving. Even in the pmp setting, our proof provides a new descriptive set theoretic understanding of ergodicity of graphs, involving a new graph invariant, a method of producing finite equivalence subrelations with large domain, and a simple method of exploiting nonamenability of a measured graph.

Furthermore, the generalization to the quasi-pmp setting turns out not to be as straightforward as one might guess. The main reason is that the proof in the invariant case uses partitions of $X$ into finite $G$-connected sets $U$ of arbitrarily large “mass” (which coincides with size in the invariant case), but in the quasi-invariant setting, the notion of “mass” is relative to a point of reference (in the same connected component) and the relative mass is given by a Borel cocycle $\rho : E_G \to \mathbb{R}^+$. The argument requires that the mass of $U$ is sufficiently large relative to every point in $U$. Thus, we define and study the notions of $\rho$-ratio (see Subsection 3.B.4) and $(G,\rho)$-visibility (Section 8), which appear to be interesting in their own right. The latter also provides a sufficient condition for hyperfiniteness (Theorem 8.4), which is used in our proof as well.

In the section following Introduction, we sketch the proof of Theorem 1.1 hoping to convey its main ideas.

**Other results.** Here, we briefly mention other results obtained in the current paper that are used in the proof of the main theorem, but are also interesting in their own right. Below, let $G$ be a locally countable Borel graph on a standard Borel space $X$.

**Cuts and hyperfiniteness.** Here, we equip $X$ with a Borel probability measure $\mu$.

**Definition 1.14.** Call a set $V \subseteq X$ a hyperfinitizing (resp. finitizing) vertex-cut for $G$ if $G-V$ is hyperfinite (resp. component-finite). We call the following quantity the hyperfinitizing vertex-price of $G$:

$$hvp_{\mu}(G) := \inf \{ \mu(C) : C \subseteq X \text{ is a Borel hyperfinitizing vertex-cut for } G \}.$$  

We also define the analogous notions for edge-cuts. This was already done in [MT17, Section 9] as well as earlier in [Ele07] in a slightly different context. The following is a useful and easily applicable way of exploiting the nonhyperfiniteness of a measurable graph and variations of it have appeared in the aforementioned two papers:

**Proposition 1.15 (Characterization of $\mu$-hyperfiniteness).** $G$ is $\mu$-hyperfinite if and only if $hvp_{\mu}(G) = 0$.

---

7For a set $V \subseteq X$, denote $G-V := G \cap (X \setminus V)^2$. 

7
Thus, for a \( \mu \)-nonhyperfinite graph \( G \), if one has built an object that works for a set of points that which form a hyperfinishing vertex-cut, then it works with probability at least \( \text{hvp}_\mu(G) > 0 \). The fact that this lower bound is independent of the construction proves to be useful in iterative arguments.

**Saturated and packed prepartitions.** When building finite subequivalence relations of a countable Borel equivalence relation \( E \) on \( X \), it is convenient to think in terms of prepartitions, i.e. collections of pairwise disjoint subsets of \( X \) (which may not cover all of \( X \)).

Letting \( [X]_E^{<\infty} \) denote the standard Borel space of \( E \)-related nonempty finite sets, we need methods of building a Borel prepartition \( \mathcal{P} \subseteq [X]_E^{<\infty} \) with a large domain\(^8\) whose each cell \( U \in \mathcal{P} \) satisfies a given Borel condition, i.e. belongs to a Borel subset \( S \subseteq [X]_E^{<\infty} \).

Due to [KM04, Lemma 7.3], there is always a Borel prepartition \( \mathcal{P} \) that is maximal within \( S \), however, in many situations (such as in our proofs), \( \text{dom}(\mathcal{P}) \) may be uncontrollably small. Thus, we need more than just maximal prepartitions, namely, we need a prepartition \( \mathcal{P} \) whose individual cells are maximally big (call such a \( \mathcal{P} \) saturated within \( S \)) and more importantly, finitely many of these cells cannot be combined together with proportionally-many points from outside of \( \text{dom}(\mathcal{P}) \) to form a good cell. Such \( \mathcal{P} \) is called packed, or more generally, \( p \)-packed, where \( p \in (0, 1] \) is the proportion parameter. The smaller the \( p \), the more packed \( \mathcal{P} \) is.

The existence of such prepartitions, modulo an \( E \)-compressible set, was proven in [MT17, Subsection 4.D] and here, we generalize this to the quasi-pmp setting in the following sense. Note that in the quasi-pmp setting, i.e. in the presence of a Borel cocycle \( \rho : E \to \mathbb{R}^+ \), \( E \)-compressible sets may have positive measure, so we replace them with the so-called \( \rho \)-deficient sets, which are necessarily \( \mu \)-null for any \( \rho \)-invariant probability measure \( \mu \).

Denoting by \( [X]_E^{\rho<\infty} \) the standard Borel space of \( E \)-related nonempty \( \rho \)-finite sets, we prove, for each Borel \( S \subseteq [X]_E^{\rho<\infty} \), the existence of a saturated and packed prepartition within \( S \) modulo a \( \rho \)-deficient set and under a mild assumption on \( S \), which holds vacuously when \( \rho \equiv 1 \). The precise definitions and statements are given in Sections 6 and 7.

**Finite cocycle-visibility and hyperfiniteness.** Let \( G \) be a locally countable Borel graph on \( X \) and let \( \rho : E_G \to \mathbb{R}^+ \) be a Borel cocycle. A \((G, \rho)\)-visible neighborhood of \( x \in X \) is any \( G \)-connected set \( V \ni x \) such that \( \rho(x, v) \geq 1 \) (i.e. the “mass” of \( x \) is at least as much as that of \( v \)) for each \( v \in V \). Denoting by \( B^G_\rho(x) \) the maximal \((G, \rho)\)-visible neighborhood of \( x \), we say that \( G \) has finite \( \rho \)-visibility if for each \( x \in X \), \( B^G_\rho(x) \) is \( \rho \)-finite. This provides a sufficient condition for Borel hyperfiniteness (see Theorem 8.4): if \( G \) has finite \( \rho \)-visibility, then \( G \) is hyperfinite.

**Cocycle-ratio and arbitrarily large sets.** A Borel cocycle \( \rho : E \to \mathbb{R}^+ \) on a countable Borel equivalence relation \( E \) provides relative measures on each \( E \)-class, which in general do not arise from an absolute measure. However, we often need to build finite subequivalence relations with arbitrarily large classes, where the largeness should somehow be with respect to \( \rho \). To make sense of this, for a nonempty \( E \)-related set \( U \subseteq X \), we define its \( \rho \)-ratio \( \rho^\text{max}(U) \) as the ratio of total \( \rho \)-mass of \( U \) and the largest \( \rho \)-mass of its individual elements. This definition makes sense by the cocycle identity and it is an absolute number attached to \( U \). However, \( \rho^\text{max} \) is not monotone or finitely additive, which makes it harder to treat it.

\(^8\)The domain of a prepartition \( \mathcal{P} \) is its union, i.e. \( \text{dom}(\mathcal{P}) := \bigcup \mathcal{P} \).
as a replacement for size. For instance, given a graph \( G \) as above, it is not hard to obtain a prepartition of an \( E_G \)-cocompressible subset of \( X \) into arbitrarily large \( G \)-connected sets: one has to take a saturated prepartition as is done in [MT17, Proof of Proposition 8.8]. Replacing size with \( \rho_{\max} \), we are still able to prove an analogous statement, but the proof is surprisingly more difficult; in particular, assuming that \( G \) is not \( \mu \)-hyperfinite, we use packed prepartitions and the fact that \( \hvp\mu(G) > 0 \). We roughly state it here, denoting by \([X]_{\rho<\infty}^G\) the standard Borel space of \( G \)-connected \( \rho \)-finite nonempty sets and referring the reader to Lemma 8.14 for the precise statement and proof.

**Lemma 1.16.** Let \( X, G, \rho, \mu \) be as above and let \( S \subseteq [X]_{\rho<\infty}^G \) be a Borel collection of sets with a mild closure condition. If \( G \) is nowhere \( \mu \)-hyperfinite\(^9\), then for every \( \varepsilon, L > 0 \), there is a Borel prepartition \( \mathcal{P} \subseteq S \) with \( \mu(\text{dom}(\mathcal{P})) \geq 1 - \varepsilon \), which contains only sets whose \( \rho_{\max} \) is at least \( L \).

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**Organization.** In Section 2 we give a sketch of the proof of our main result, Theorem 1.1. Section 3 establishes notation and terminology that are globally used in the paper. In Section 4 we briefly discuss finite and hyperfinite averages, and in particular, state the general version of Theorem 1.6. Section 5 provides a proof of the equivalence of Theorems 1.1, 2.1, 1.4 and 1.7. In Section 6 we discuss flows along a cocycled equivalence relation, introduce the notion deficiency for sets as a replacement for compressibility, and provide a lemma for building Borel flows. Section 7 introduces saturated and packed prepartitions with respect to a cocycle and proves their existence. Section 8 discusses the notion of cocycle-visibility on a graph, provides a sufficient condition for hyperfiniteness (Theorem 8.4), and proves the lemma on partitioning into \( \rho_{\max} \)-large sets (Lemma 8.14). In Section 9 we introduce an invariant for a graph called the set of asymptotic averages, whose role is instrumental for the proof of our main result; we then establish a local-global correspondence between the set of asymptotic averages and existence of certain partitions (Corollary 9.11). Section 10 discusses cuts and their connection with hyperfiniteness. Finally, Section 11 puts everything together into a proof of Theorem 1.1.

## 2. Sketch of proof

Diagonalizing against a countable dense subset of \( L^1(X, \mu) \), Theorem 1.1 boils down to the following (see Section 5 for details):

\(^9\)This means that there is no \( E_G \)-invariant Borel set of positive \( \mu \)-measure on which \( G \) is hyperfinite.
We say that a set \( S \subseteq X \) is \( \mu\text{-co-} \varepsilon \) if \( \mu(S) \geq 1 - \varepsilon \).
we have is that \( \int_X f \, d\mu = 0 \), which is global, whereas \( A_f[G] \) is defined locally. So we need a local-global correspondence lemma that would yield a global consequence from the properties of \( A_f[G] \). Here it is:

(2.4) **Uniformly achieving some asymptotic averages.** For every \( \delta > 0 \) there is a finite Borel \( G \)-connected equivalence relation \( F \) such that \( A_f[F](x) \in_\delta A_f[G] \) for a.e. \( x \in X \).

Thus, if \( 0 \notin_\varepsilon A_f[G] \), then by convexity (2.3), \( A_f[G] \) completely lies on one side of \( I_\varepsilon \) in \( \mathbb{R} \), so an \( F \) from (2.4) will move the global mean of \( f \) away from 0 because

\[
\int_X f \, d\mu = \int_X A_f[F] \, d\mu.
\]

In fact, denoting by \( A_f[G:F] \) the set of asymptotic averages of the quotient by \( F \) (replacing \( G \) with \( G/F \) and \( f \) with \( A_f[F] \)), a quantitative refinement of the last argument allows us to assume the following without loss of generality:

(2.5) **Hypothesis.** There is a \( \delta > 0 \) such that for any \( G \)-connected finite Borel equivalence relation \( F, A_f[G:F] \) spills over both sides of \( I_\delta \), i.e. meets both \( I_- \delta := (-\infty, -\delta] \) and \( I_+ \delta := [\delta, +\infty) \).

2.B. **Packed prepartitions and finitizing cuts**

For this step we use our results from Sections 7 and 10, which are also briefly discussed in the “Other results” part of Introduction.

To define a finite equivalence relation \( F \) as in the conclusion of Theorem 2.1, we need to define a prepartition \( P \) with \( \mu \)-co-\( \varepsilon \) domain, whose each cell \( U \in P \) is \( G \)-connected and \( A_f[U] \approx_\varepsilon 0 \); we temporarily call such a set \( U \) good.

![Figure 1. A maximal prepartition \( P \) whose cells are depicted as rectangles](image)

As a first attempt to build such a \( P \), we may take a Borel prepartition that is maximal within the collection of good sets; such a \( P \) exists by [KM04, Lemma 7.3]. But how large would \( \text{dom}(P) \) be? Because \( 0 \notin A_f[G] \) by (2.5), \( \text{dom}(P) \) would have to meet every \( G \)-connected component, so it would be of positive measure, but this measure can be arbitrarily small. Indeed, each \( G \)-connected component could have gigantic (infinite) connected “continents” (sets) on which \( f \) was either very positive (> 100) or very negative (< -100) and they were linked to each other by small “islands” on which \( f \) was 0; \( P \) could have picked up these “islands” and left out the “continents”. See Fig. 1.
To force the prepartition \( \mathcal{P} \) to put parts of the positive and negative continents into a cell, we need to require more than just maximality from \( \mathcal{P} \), namely, we need a prepartition \( \mathcal{P} \) whose individual cells are maximally good (call such a \( \mathcal{P} \) saturated) and more importantly, finitely many of these cells cannot be combined together with proportionally-many points from outside of \( \text{dom}(\mathcal{P}) \) to form a good cell. Thus, we let \( \mathcal{P} \) be a packed prepartition, whose existence modulo an \( E_G \)-compressible (and hence null) set is proven in Theorems 7.9 and 7.16.

![Figure 2. Forming a pack over a prepartition \( \mathcal{P} \)](image)

Under hypothesis (2.5), packed prepartitions do not leave out infinite “continents”. Indeed, Fig. 1 is no longer possible because one could form a good cell (a pack) as in Fig. 2, taking proportionally-many points outside of \( \text{dom}(\mathcal{P}) \), which contradicts the packedness of \( \mathcal{P} \).

(2.6) **Prepartitions with finitizing domain.** Let \( \delta \) be as in (2.5), let \( \lambda, L > 0 \) with \( \lambda < \delta \), and put \( p := \frac{\lambda}{\|f\|_{\infty}} \). For any prepartition \( \mathcal{P} \) that is \( p \)-packed within the collection of all \( U \in [X]_{G}^{\infty} \) with \( A_f[U] \in I_{\lambda} \) and \( |U| \geq L \), \( G_{-\text{dom}(\mathcal{P})} \) is component-finite.

Thus, for a \( \mathcal{P} \) as in (2.6), its domain is a finitizing vertex-cut for \( G \) and hence, the non-\( \mu \)-hyperfiniteness of \( G \) implies that \( \mu(\text{dom}(\mathcal{P})) \geq \text{hvp}_\mu(G) > 0 \), where we recall that \( \text{hvp}_\mu(G) \) is the infimum of the measures of all hyperfinitizing vertex-cuts for \( G \). Of course, \( \text{hvp}_\mu(G) \) may still be very small, whereas we want \( 1 - \epsilon \). However, it is an absolute lower bound, i.e. depends only on \( G \), which makes an iterative construction of a desired preparation possible.

2.C. **Iteration via measure-compactness**

We take a coherent sequence \((\mathcal{P}_n)\) of \( G \)-prepartitions that get more and more packed and contain larger and larger sets whose \( f \)-averages get closer and closer to 0. By coherent we only mean that each cell \( U \in \mathcal{P}_n \) doesn’t break any cells in \( \bigcup_{k<n} \mathcal{P}_k \), i.e. \( U \) is \( E_{\mathcal{P}_k} \)-invariant\(^{11}\) for all \( k < n \). This ensures that the union of the induced equivalence relations \( E(\mathcal{P}_n) \) is an equivalence relation, in particular, \( F_{\infty} := \bigcup_n E(\mathcal{P}_n) \) is a hyperfinite equivalence relation, but it does not imply that the domains \( D_n := \text{dom}(\mathcal{P}_n) \) are increasing or even have

\(^{11}\) For a prepartition \( \mathcal{P} \), \( E(\mathcal{P}) \) denotes the induced equivalence relation on \( X \), i.e. its classes are the cells in \( \mathcal{P} \) and singletons outside of \( \text{dom}(\mathcal{P}) \).
a nonempty intersection, neither does it imply that their union has large measure. We let
\( D_\infty \subseteq X \) denote the set where each \( F_\infty \)-class is infinite, equivalently,
\[
D_\infty := \limsup_n D_n := \{ x \in X : \exists n x \in D_n \} = \bigcap_{n \geq N} \bigcup_{n \geq N} D_n.
\]

Recalling that \( \mu(D_n) \geq hvp_\mu(G) > 0 \), the finiteness of \( \mu \) implies (see Observation 11.8):

(2.7) Combining the prepartitions. \( D_\infty \) has measure at least \( hvp_\mu(G) > 0 \).

All we need actually is that \( D_\infty \) has positive measure because then it meets a.e. \( G \)-connected component and in each of them the saturation and packedness of the \( \mathcal{P}_n \) affirm that the points outside of \( D_\infty \) had no good reason to not be involved into \( \mathcal{P}_n \) for all large enough \( n \), so there aren’t such points:

(2.8) Covering the space. \( D_\infty \) is conull.

Thus, for a large enough \( n \), taking \( F := \bigcup_{k<n} E(\mathcal{P}_k) \) satisfies the conclusion of Theorem 2.1.

2.D. The general quasi-invariant setting

Assuming that \( \mu \) is only \( E_G \)-quasi-invariant, we let \( \rho : E_G \rightarrow \mathbb{R}^+ \) be the corresponding Radon–Nikodym cocycle. For points \( x, y \in X \) in the same \( E_G \)-class, we think of the value \( \rho(x,y) \) as the ratio of the masses of \( x \) and \( y \), or rather, the mass of \( x \) relative to \( y \). To emphasize this intuition, we write \( \rho^y(x) \) in lieu of \( \rho(x,y) \) and think of \( \rho^y \) as a distribution/measure on \( [y]_G \).

Here we simply list the main changes and additional considerations one has to make for the argument above to go through.

(2.9) \( E_G \)-compressible \( \leadsto \) \( \rho \)-deficient. The notion of an \( E_G \)-compressing map is replaced with that of a \( \rho \)-flow with no sinks but lots of sources, whose definition (Definition 6.4) is basically equivalent to that of \( \rho \)-invariant fuzzy partial injection defined in [Mil08]. The sets on which such flows exist are called \( \rho \)-deficient and they are necessarily \( \mu \)-null for any \( \rho \)-invariant probability measure \( \mu \).

(2.10) \( G \)-asymptotic averages \( \leadsto \) \((G,\rho)\)-visible asymptotic averages. The convexity of \( \mathcal{A}_f[G] \) is crucially used in the argument above, but its analogue in the presence of \( \rho \) may not be convex if \( \rho^x \) is unbounded\(^{12}\) on a given \( E_G \)-class \( [x]_G \). Indeed, when taking larger and larger connected sets containing \( x \), we may encounter arbitrarily \( \rho^x \)-large points with very positive and very negative values of \( f \), which can easily result in \(-100\) and \(100\) being \( G \)-asymptotic averages, without \( 0 \) being one. Thus, to preserve convexity, we only allow taking arbitrarily large connected sets within a \( \rho^x \)-bounded set \( B \), namely, \( B := \{ y \in [x]_G : \rho^x(y) \leq \alpha \} \), for some fixed constant \( \alpha \in [1,\infty) \). Thinking of points in \( B \) as being visible from \( x \) (with \( \alpha \)-magnification), we call these \((G,\rho)\)-visible asymptotic averages and denote their set by \( \mathcal{A}_f^G[x] \). The analogues of (2.2) and (2.3) now hold.

(2.11) Size of a finite set \( \leadsto \) its \( \rho \)-ratio. Proving (2.4) for \( \mathcal{A}_f^G[G] \) in lieu of \( \mathcal{A}_f[G] \) gets much trickier. The finite Borel equivalence relation \( F \) in (2.4) is induced by taking any saturated prepartition within the collection of \( G \)-connected finite sets \( U \) of large enough size. To ensure that saturation does the same for \((G,\rho)\)-visible asymptotic

\(^{12}\)This only depends on \([x]_G\) and not on \( x \).
averages, we need to replace size with ρ-ratio, as discussed in the “Other results” of Introduction:

\[ \rho_{\text{max}}(U) := \frac{\rho^x(U)}{\max_{u \in U} \rho^x(u)}. \]

The utility of \( \rho_{\text{max}}(U) \) over \( \rho^x \) is that it does not depend on the choice of the reference point \( x \in [U]_{E_G} \); however \( \rho_{\text{max}}(U) \) is much harder to work with as it is not monotone or additive. See Definition 3.5, Lemma 8.14, and Corollary 9.11.

3. Preliminaries

Throughout, let \( X \) be a standard Borel space.

3.A. Equivalence relations

Let \( E \) denote a countable Borel equivalence relation on \( X \), where “countable” refers to the size of each \( E \)-class, of course. We refer to [JKL02] and [KM04] for the general theory of countable equivalence relations.

We say that a set \( A \subseteq X \) is \( E \)-related if it is contained in a single \( E \)-class; similarly, we say that points \( x_0, x_1, \ldots, x_n \in X \) are \( E \)-related if \( \{x_0, x_1, \ldots, x_n\} \) is \( E \)-related. As usual, we denote by \( [X]^{<\infty}_E \) the standard Borel space of finite nonempty \( E \)-related sets.

When \( E \) is smooth, i.e. admits a Borel reduction to the identity equivalence relation \( \text{Id}_R \) on \( R \), the quotient space is also standard Borel and we denote by it by \( X/E \).

Definition 3.1. For a Borel measure \( \mu \) on \( X \), we say that \( E \) is measure preserving (mp) or that \( \mu \) is \( E \)-invariant if for every Borel automorphism \( \varphi \) of \( E \), \( \varphi^* \mu = \mu \). More generally, we say that \( E \) is quasi measure preserving (quasi-mp) or that \( \mu \) is \( E \)-quasi-invariant if for every Borel automorphism \( \varphi \) of \( E \), \( \varphi^* \mu \sim \mu \). We call \( E \text{-mp} \) (resp. quasi-\( E \text{-mp} \)) if it is mp (respectively, quasi-mp) and \( \mu \) is a probability measure.

3.B. Cocycles

For a countable Borel equivalence relation \( E \) on \( X \), a cocycle on \( E \) is a map \( \rho : E \rightarrow \mathbb{R}^+ \) satisfying the cocycle identity: \( \rho(x, y) \cdot \rho(y, z) = \rho(x, z) \), for any \( E \)-related \( x, y, z \in X \). Below, we let \( \rho \) denote a Borel cocycle on \( E \).

3.B.1. Cocycle as mass-ratio. For \( E \)-related \( x, y \in X \), we think of the value \( \rho(x, y) \) as

\[ \text{mass}(x)/\text{mass}(y), \]

or rather, the mass of \( x \) relative to \( y \). To emphasize this intuition, for each \( E \)-class \( Y \) and a “reference point/origin” \( o \in Y \), we define a distribution \( \rho^o : Y \rightarrow \mathbb{R}^+ \) by \( \rho^o(y) := \rho(y, o) \). We also denote by \( \rho^o \) the corresponding measure on \( Y \), so for a set \( A \subseteq Y \),

\[ \rho^o(A) := \sum_{y \in A} \rho^o(y). \]

and for a function \( f : Y \rightarrow \mathbb{R}^+ \),

\[ \int_Y f \, d\rho^o := \sum_{y \in Y} f(y) \rho^o(y). \]
Note that for any \( x, y \in Y \), the value of ratios of homogeneous linear polynomials with variables of the form \( \rho^o(\cdot) \) or \( \int_Y f d\rho^o \), such as \( \frac{2\rho^o(y) + \int_Y f d\rho^o}{\rho^o(A)} \), does not depend on the choice of the origin \( o \). Similarly, the validity of homogeneous statements like "\( \rho^o(A) \) is finite" or

\[
2 \cdot \rho^o(y) + \int_Y f d\rho^o \geq \rho^o(A)
\]

is also independent of \( o \). We refer to these as \( \rho \)-homogeneous expressions and omit the superscript \( o \) from them. Thus, it makes sense to write \( \rho(x, y) = \rho(x)/\rho(y) \).

3.B.2. **Quasi-invariant measures.** For a Borel cocycle \( \rho : E \to \mathbb{R}^+ \), a Borel measure \( \mu \) on \( X \) is called \( \rho \)-invariant if for every Borel set \( B \subseteq X \) and a Borel automorphism \( \varphi \) of \( E \),

\[
\mu(\varphi(B)) = \int_B \rho^\varphi(\varphi(x)) d\mu(x).
\]

Because such a cocycle is clearly unique, we also call \( \rho \) the cocycle corresponding to \( \mu \). It is a standard fact (see, for example, [KM04, Section 8]) that every \( E \)-quasi-invariant probability measure \( \mu \) has a corresponding Borel cocycle \( \rho : E \to \mathbb{R}^+ \).

For a smooth Borel subequivalence relation \( F \subseteq E \), we let \( \mu_F \) denote the quotient measure on \( X/\mathcal{F} \). Clearly, \( \mu_F \) is \( E/\mathcal{F} \)-quasi-invariant and its corresponding cocycle is \( \rho_F : E/\mathcal{F} \to \mathbb{R}^+ \) defined by \( \rho_F([x]_\mathcal{F},[y]_\mathcal{F}) = \rho([x]_\mathcal{F})/\rho([y]_\mathcal{F}) \).

3.B.3. **The space \([X]_E^{\rho<\infty}\) of \( \rho \)-finite sets.**

**Definition 3.2.** An \( E \)-related set \( A \) is said to be \( \rho \)-finite if \( \rho(A) \) is finite, otherwise, it is \( \rho \)-infinite. We denote by \([X]_E^{\rho<\infty}\) the collection of all nonempty \( E \)-related \( \rho \)-finite sets.

We show that \([X]_E^{\rho<\infty}\) is also a standard Borel space; in fact, it can be naturally viewed as a Borel subset of \( X^{[\mathbb{N}]} : = X^{\mathbb{N}} \cup X^{\mathbb{N}} \).

Throughout, we fix a Borel linear order \( \langle \rho \) on \( X \) and define a linear order \( <_\rho \) between any two \( E \)-related elements \( x, y \in X \) as follows:

\[
x <_\rho y \Leftrightarrow \rho(x) < \rho(y) \text{ or } \left( \rho(x) = \rho(y) \text{ and } x <_\chi y \right).
\]

**Observation 3.3.** For each \( A \in [X]_E^{\rho<\infty} \) and \( a_0 \in A \), the set \( \{ a \in A : a >_\rho a_0 \} \) is finite.

This implies:

**Corollary 3.4.** The relation \( >_\rho \) on \( A \in [X]_E^{\rho<\infty} \) is a well-ordering of type \( \leq \omega \).

Thus, we identify \([X]_E^{\rho<\infty}\) with the set of all \( <_\rho \)-decreasing \( E \)-related sequences of \( X \) of length \( \leq \omega \), which is clearly a Borel subset of \( X^{<\infty} \). Furthermore, we view \([X]_{E}^{\rho<\infty}\) as a Borel subset of \([X]_E^{\rho<\infty}\).

3.B.4. **\( \rho \)-ratio of \( E \)-related sets.** Because it is typically impossible to select an origin from each \( E \)-class in a Borel, there isn’t, in general, a Borel assignment of a distribution of the form \( \rho^o \) to each \( E \)-class. In fact, such an assignment exists exactly when \( \rho \) is a coboundary.
While in our arguments below we do not need an absolute notion of mass on \([X]^0_{E}\), we still need a Borel function \(r : [X]^0_{E} \to \mathbb{R}^+\) such that
\[
U \subseteq V \implies \frac{r(V)}{r(U)} \leq \frac{\rho(V)}{\rho(U)}.
\]
for any \(U, V \in [X]^0_{E}\). A candidate for such a function is the following: take any Borel function
\[
s : [X]^0_{E} \to X
\]
that assigns a point in \([U]_E\) to each \(U \in [X]^0_{E}\) and define a function
\[
\rho^s : [X]^0_{E} \to \mathbb{R}^+ \text{ by } U \mapsto \rho^s(U).
\]
This function would serve its purpose if \(s\) is \(\rho\)-monotone, i.e. \(U \subseteq V \implies \rho(s(U)) \leq \rho(s(V))\).
But such is the most natural selector function \(\max_{\rho} : [X]^0_{E} \to X\) mapping each \(U \in [X]^0_{E}\) to its \(\rho\)-largest element.

**Definition 3.5.** For each \(U \in [X]^0_{E}\), its \(\rho\)-ratio is the quantity
\[
\rho^\max(U) := \frac{\rho(U)}{\max_u \rho(u)}.
\]

**Observation 3.6.** For any \(U, V \in [X]^0_{E}\), \(U \subseteq V\) implies \(\rho^\max(V) \leq \frac{\rho(V)}{\rho(U)}\). In particular, for any increasing sequence \((U_n)\) of sets in \([X]^0_{E}\), \(\rho^\max(U_n) \to \infty\) implies \(\rho(U_n) \to \infty\).

Lastly, we say that an \(E\)-related set \(A\) is \(\rho\)-bounded if for some (equivalently, any) origin \(o \in [A]_E\), the function \(\rho|_A\) is bounded. We extend \(\rho^\max\) to \(\rho\)-infinite and \(\rho\)-bounded sets by declaring it to be \(\infty\) on such sets.

### 3.C. Graphs
By a Borel graph on \(X\) we mean an irreflexive and symmetric Borel subset of \(X^2\). In this paper, we only consider *locally countable* Borel graphs, i.e. the degree of each vertex is countable. Let \(G\) denote such a graph.

For sets \(A, B \subseteq X\), the following is standard notation:
- \([A, B] := (A \times B \cup B \times A) \setminus \text{Id}_X\);
- \([A, B]_G := [A, B] \cap G\);
- \(G|_A := G \cap A^2\);
- \(\partial^\text{out}_G A := \{ x \in X \setminus A : \exists a \in A \ (x, a) \in G\} \) — we refer to this as the *outer G-boundary* of \(A\).

We denote by \(E_G\) the equivalence relation of being in the same \(G\)-connected component and for convenience we denote the \(E_G\)-class of \(x \in X\) by \([x]_G\). Furthermore, we call a set \(U \subseteq X\) \(G\)-connected if \(G \cap U^2\) is a connected graph on \(U\), and we let \([X]_{E_G}^\infty\) denote the (Borel) subset of \([X]_{E_G}^\infty\) of \(G\)-connected sets. Similarly, for a Borel cocycle \(\rho : E_G \to \mathbb{R}^+\), we let \([X]_{E_G}^\rho\) denote the corresponding subset of \([X]_{E_G}^\infty\).

We say that \(G\) is *component-finite* if \(E_G\) is finite.
**Definition 3.7.** For a Borel measure $\mu$ on $X$, we say that $G$ is **pmp**, **quasi-pmp**, **ergodic**, **hyperfinite** if so is $E_G$.

We say that an equivalence relation $F$ on $X$ is **$G$-connected** if each $F$-class is $G$-connected; in particular, $F \subseteq E_G$ but the converse does not hold in general.

Given a smooth Borel subequivalence relation $F \subseteq E_G$, we form the quotient graph $G/F$ by contracting the edges between $F$-related vertices, i.e. for $U, V \in X/F$,

$$(U, V) \in G/F :\iff [U, V]_G \neq \emptyset.$$ 

In our arguments, we need every $G/F$-connected set $A \subseteq X/F$ to lift to a $G$-connected set $A \subseteq X$; this happens exactly when $F$ is $G$-connected, so we will only take quotients of $G$ by $G$-connected equivalence relations.

4. Finite and hyperfinite averages

This subsection is the generalization of [MT17, Subsections 7.A–B] to the quasi-invariant setting.

We start with an abstract observation about finite averages.

**Lemma 4.1 (Convexity of average).** Let $Y$ be a set, $\rho : Y^2 \to \mathbb{R}^+$ be a cocycle$^{13}$, and $f : Y \to \mathbb{R}^+$. For any nonempty disjoint $\rho$-finite sets $U, V \subseteq Y$,

\begin{align*}
&\text{(4.1.a)} \quad A^\rho_f[U \cup V] = \frac{\rho(U)}{\rho(U)+\rho(V)} A^\rho_f[U] + \frac{\rho(V)}{\rho(U)+\rho(V)} A^\rho_f[V], \\
&\text{(4.1.b)} \quad |A^\rho_f[U \cup V] - A^\rho_f[U]| \leq 2\|f\|_\infty \frac{\rho(V)}{\rho(U)+\rho(V)} \leq 2\|f\|_\infty \frac{\rho(V)}{\rho(U)}.
\end{align*}

**Proof.** One verifies (4.1.a) directly, and (4.1.b) follows from (4.1.a) by the triangle inequality. \hfill \Box

Moving back to the measurable setting, let $(X, \mu)$ be a standard probability space and fix $f \in L^1(X, \mu)$.

**Proposition 4.2.** Let $F$ be a finite quasi-pmp equivalence relation on $(X, \mu)$ with $\rho : F \to \mathbb{R}^+$ being the corresponding cocycle.

\begin{align*}
&\text{(4.2.a)} \quad \int_X f \, d\mu = \int_X A^\rho_f[F] \, d\mu. \\
&\text{(4.2.b)} \quad \|A^\rho_f[F]\|_1 \leq \|f\|_1.
\end{align*}

**Proof.** (4.2.b) is immediate from (4.2.a) and the fact that $|A^\rho_f[F]| \leq A^\rho_f[F]$. As for (4.2.a), for each $n \in \mathbb{N}$, we may restrict to the part where each $F$-class has size $n$, which allows us to assume without loss of generality that for some $n \in \mathbb{N}$, all $F$-classes have size $n$. We then take a for any Borel transversal$^{14}$ $S$ through $F$ and a Borel automorphism $T \in [F]$ that

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$^{13}$Treating $Y^2$ as the trivial equivalence relation on $Y$.

$^{14}$A set $S \subseteq X$ is a **transversal** through an equivalence relation $F$ if it intersects every $F$-class in exactly one point.
induces $F$, so for every $x \in S$, $[x]_F = \{T^i x\}_{i < n}$. We compute:

$$\int_X f \, d\mu = \sum_{i < n} \int_{T^i(S)} f(x) \, d\mu(x)$$

$[x \mapsto T^i x \text{ using } \rho\text{-invariance}] = \sum_{i < n} \int_S f(T^i x) \cdot \rho^x(T^i x) \, d\mu(x)$

$$= \int_S \sum_{i < n} f(T^i x) \cdot \rho^x(T^i x) \, d\mu(x) = \int_S A_f^\rho(F)(x) \cdot \rho^x([x]_F) \, d\mu.$$

On the other hand:

$$\int_S A_f^\rho(F)(x) \cdot \rho^x([x]_F) \, d\mu = \int_S A_f^\rho(F)(x) \cdot \sum_{i < n} \rho^x(T^i x) \, d\mu(x)$$

$[A_f^\rho(F) \text{ is } T\text{-invariant}] = \sum_{i < n} \int_S A_f^\rho(F)(T^i x) \cdot \rho^x(T^i x) \, d\mu(x)$

$[x \mapsto T^{-i} x \text{ using } \rho\text{-invariance}] = \sum_{i < n} \int_{T^{-i}(S)} A_f^\rho(F)(x) \, d\mu(x) = \int_X A_f^\rho(F) \, d\mu. \quad \square$

**Lemma 4.3** (Approximate $L^\infty$-continuity). For a finite quasi-pmp Borel equivalence relation $F$ on $(X, \mu)$ and $\varepsilon > 0$, there is an $F$-invariant $\mu$-co-$\varepsilon$ set $X' \subseteq X$ such that

$$\|A_f^\rho(F)|_{X'}\|_\infty \leq \frac{1}{\varepsilon} \|f\|_1.$$

**Proof.** This is just (4.2.b) combined with Chebyshev’s inequality. \qed

The following is [MT17, Theorem 7.3] in the quasi-invariant setting, but the proof is the same, so we omit it.

**Theorem 4.4** (Pointwise ergodic theorem for hyperfinite equivalence relations). Let $F$ be a Borel quasi-pmp hyperfinite equivalence relation on $(X, \mu)$ and let $\rho : F \to \mathbb{R}^+$ be a Borel cocycle corresponding to $\mu$. For any $f \in L^1(X, \mu)$ and any witness $(F_n)$ to the hyperfiniteness of $F$, the pointwise limit

$$A_f^\rho(F) := \lim_{n \to \infty} A_f^\rho(F_n)$$

exists a.e. and is equal to the conditional expectation of $f$ with respect to the $\sigma$-algebra of $F$-invariant measurable sets, i.e.

(4.4.a) Projection onto $F$-invariant: $\int A_f^\rho(F) \cdot g \, d\mu = \int f \cdot g \, d\mu$ for any $F$-invariant $g \in L^\infty(X, \mu)$. In particular,

(4.4.b) $F$ is ergodic if and only if $A_f^\rho(F) = \int_X f \, d\mu$ a.e.

(4.4.c) $L^1$-bound: $\|A_f^\rho(F)\|_1 \leq \|f\|_1$.

We call $A_f^\rho(F)$ the $\rho$-mean of $f$ over $F$. 
5. The equivalence of the main theorems

In this section, we prove the equivalence of Theorems 1.1, 2.1, 1.4 and 5.1. The implications 1.7 $\Rightarrow$ 1.1 $\Rightarrow$ 2.1 are obvious. Furthermore, Theorem 4.4 reveals the equivalence of 1.1 and 1.4 via the standard trick of replacing an arbitrary probability measure $\mu$ with an $E_G$-quasi-invariant one, e.g., $\sum_{n \geq 1} 2^{-n} \gamma_n * \mu$, where $(\gamma_n) \subseteq [E_G]$ is a sequence of Borel automorphisms such that $E = \bigcup_n \text{Graph}(\gamma_n)$ (which exists by the Feldman–Moore theorem). For 1.1 $\Rightarrow$ 1.7, we observe that $d\mu_g := \frac{1}{\int_X gd\mu} gd\mu$ is still $E_G$-quasi-invariant and the Radon–Nikodym cocycle $\rho_g(y,x)$ of $\mu_g$ is $g(y)\rho(y,x)g(x)^{-1}$, so

$$\frac{A^\rho_{fg}[F_n]}{A^\rho_g[F_n]} = A^\rho_{fg^{-1}} \quad \text{and} \quad \frac{\int_X f d\mu}{\int_X gd\mu} = \int_X f g^{-1} d\mu_g.$$

It remains to show the implication 2.1 implies 1.1, so let $X, \mu, G, \rho$ be as in the latter theorem. Let $\mathcal{D} \subseteq L^1(X, \mu)$ be countable dense and let $(f_n)$ be an enumeration of $\mathcal{D}$ such that each $f \in \mathcal{D}$ is equal to $f_n$ for infinitely-many $n \in \mathbb{N}$. Furthermore, let $(\varepsilon_n)$ be a sequence of reals converging to 0. Applying 2.1 to the functions $f_n$ over finite quotients, we iteratively build an increasing sequence $(F_n)$ of $G$-connected finite Borel equivalence relations such that for each $n$, $A^\rho_{f_n}[F_n] \approx_{\varepsilon_n} \int_X f_n d\mu$ on an $F_n$-invariant $\mu$-co-$\varepsilon_n$ set $X_n \subseteq X$. Now fix $f \in L^1(X, \mu)$. By Theorem 4.4, it is enough to extract a subsequence $(F_{n_k})$ such that

$$\lim_{n \to \infty} A^\rho_{f}[F_{n_k}] = \int_X f d\mu \quad \text{for a.e. } x \in X,$$

by the independence of the witness to the hyperfiniteness of $F := \bigcup_n F_n$. By the Borel–Cantelli lemma, it is enough to show that for any $\epsilon > 0$, there is $n \in \mathbb{N}$ and an $F_n$-invariant $\mu$-co-$\varepsilon$ set $X' \subseteq X$ such that $A^\rho_{f}[F_{n_k}] \approx_{\epsilon} \int_X f d\mu$ for all $x \in X'$. To this end, we fix $\epsilon > 0$ and let $n \in \mathbb{N}$ be such that $\epsilon_n$ and $\|f - f_n\|_1$ are both less than $\frac{\epsilon^2}{4}$. By Lemma 4.3,

$$\left| A^\rho_{f_n}[F_n] - A^\rho_{f}[F_n] \right| \leq A^\rho_{|f_n - f|}[F_n] \leq \frac{2}{\epsilon} \|f_n - f\|_1 < \frac{\epsilon}{2},$$

on a $\mu$-co-$\varepsilon$ set. Thus, $A^\rho_{f}[F_n] \approx_{\epsilon} \int_X f d\mu$ on a $\mu$-co-$\varepsilon$ set.

6. Flows along a cocycle

Throughout this section, let $X$ be a standard Borel space, $E$ be a countable Borel equivalence relation on $X$, and $\rho : E \to \mathbb{R}^+$ a Borel cocycle.

6A. Preliminaries

For a function $\varphi : E \to [0, \infty)$, we think of the value $\varphi(x, y)$ as the fraction of the $\rho$-mass of $x$ that flows from $x$ to $y$; thus, what $y$ receives in this transaction is $\varphi(x, y)\rho^y(x)$. With this in mind, we define functions $\text{in}^\rho \varphi$, $\text{out}^\rho \varphi : X \to [0, \infty]$ by setting, for $x, y \in X$,
\[ \text{out}^\rho \varphi(x) := \sum_{y \in [x]_E \setminus [x]} \varphi(x, y) \]
\[ \text{in}^\rho \varphi(y) := \sum_{x \in [y]_E \setminus [y]} \varphi(x, y) \cdot \rho^y(x). \]

**Definition 6.1.** We call a function \( \varphi : E \to [0, \infty) \) a \( \rho \)-flow if \( \text{out}^\rho \varphi \) and \( \text{in}^\rho \varphi \) are bounded by 1.

The definition of a \( \rho \)-flow is exactly the same as that of a \( \rho \)-invariant fuzzy partial injection defined in [Mil08].

When defining a flow \( \varphi \) below, we will only partially specify its values with the convention that the undefined values are treated as 0; thus, it makes sense to define the domain of \( \varphi \) as follows:
\[ \text{dom}(\varphi) := \{(x, y) \in E : \varphi(x, y) \neq 0\}. \]

We further define:
- The vertex-domain of \( \varphi \), noted \( \text{vdom}(\varphi) \), is defined to be the set \( \text{proj}_0(\text{dom}(\varphi)) \cup \text{proj}_1(\text{dom}(\varphi)) \).
- For sets \( Y, Z \subseteq X \), we say \( Y \times Z \) is \( \varphi \)-closed if \( \text{dom}(\varphi) \) is disjoint from \( (Y \times Z^c) \cup (Y^c \times Z) \).
- We say that a \( \rho \)-flow \( \varphi' \) extends a \( \rho \)-flow \( \varphi \), noted \( \varphi' \subseteq \varphi \), if \( \text{dom}(\varphi') \supseteq \text{dom}(\varphi) \) and their values coincide on \( \text{dom}(\varphi) \).
- Put \( \partial^\rho \varphi := \text{in}^\rho \varphi - \text{out}^\rho \varphi \) and call it the net flow of \( \varphi \).
- Lastly, we call \( \varphi \) the zero \( \rho \)-flow if \( \text{dom}(\varphi) = \emptyset \).

### 6.B. Flow in equals flow out

**Lemma 6.2.** For any sets \( U, V \subseteq X \) such that \( U \cup V \) is \( E \)-related, if \( U \times V \) is \( \varphi \)-closed, then
\[ \int_U \text{out}^\rho \varphi \, d\rho = \int_V \text{in}^\rho \varphi \, d\rho. \]

**Proof.** This is just the Fubini theorem for finite sums:
\[
\int_U \text{out}^\rho \varphi \, d\rho = \sum_{u \in U} \text{out}^\rho \varphi(u) \cdot \rho(u)
= \sum_{u \in U} \sum_{v \in V \setminus \{u\}} \varphi(u, v) \cdot \rho(u)
= \left[ \text{Fubini} \right] \sum_{v \in V} \sum_{u \in U \setminus \{v\}} \varphi(u, v) \cdot \frac{\rho(u)}{\rho(v)} \cdot \rho(v)
= \sum_{v \in V} \text{in}^\rho \varphi(v) \cdot \rho(v) = \int_V \text{in}^\rho \varphi \, d\rho. \qed
\]

The following is [Mil08, Proposition 6.4] adapted to our terminology.

**Proposition 6.3.** For a \( \rho \)-invariant Borel probability measure \( \mu \) on \( X \) and a Borel \( \rho \)-flow \( \varphi \),
\[ \int_X \partial^\rho \varphi \, d\mu = 0. \]
Proof. The proof is just the uniformization of that of Lemma 6.2. By the Feldman–Moore theorem, \(E \setminus \text{Id}_X\) is disjoint union \(\bigsqcup_n \text{Graph}(\gamma_n) \setminus \text{Id}_X\), where each \(\gamma_n\) is a Borel involution of \(X\). Thus, putting \(X_n := \{x \in X : \gamma_n(x) \neq x\}\), we compute

\[
\int_X \text{out}^\rho \varphi(x) d\mu(x) = \int_{X \times X} \sum_{n \in \mathbb{N} : \gamma_n(x) \neq x} \varphi(x, \gamma_n(x)) d\mu(x)
\]

\[
\text{[Fubini]} = \sum_n \int_{X_n} \varphi(x, \gamma_n(x)) d\mu(x)
\]

\[
\text{[change of variable } x \mapsto \gamma_n^{-1}(x) = \gamma_n(x)] = \sum_n \int_{X_n} \varphi(\gamma_n(x), x) \cdot \frac{\rho(\gamma_n(x))}{\rho(x)} d\mu(x)
\]

\[
\text{[Fubini]} = \int_{X \times X} \sum_{n \in \mathbb{N} : \gamma_n(x) \neq x} \varphi(\gamma_n(x), x) \cdot \frac{\rho(\gamma_n(x))}{\rho(x)} d\mu(x)
\]

\[
= \int_X \text{in}^\rho \varphi(x) d\mu(x). \quad \square
\]

6.C. Deficiency and \(\rho\)-invariant measures

Definition 6.4. Let \(\varphi\) be a \(\rho\)-flow.

• Call a point \(x \in X\) a \(\varphi\)-source (resp. \(\varphi\)-sink) if \(\partial^\rho \varphi(x)\) is negative (resp. positive). We denote the sets of \(\varphi\)-sources and \(\varphi\)-sinks by \(\text{Sources}(\varphi)\) and \(\text{Sinks}(\varphi)\), respectively.

• We say that \(\varphi\) disbalances an \(E\)-class \(C\) if \(C\) contains \(\varphi\)-sources but no \(\varphi\)-sinks or vice versa, \(\varphi\)-sinks but no \(\varphi\)-sources. We say that \(\varphi\) disbalances an \(E\)-invariant set \(Z \subseteq X\) if it disbalances every \(E\)-class of \(Z\).

• Call \(Y \subseteq X\) \(\rho\)-deficient if there is a Borel \(\rho\)-flow \(\varphi\) disbalancing \([Y]_E\).

Taking sums of \(\rho\)-flows, we see that:

Observation 6.5. \(\rho\)-deficient sets form a \(\sigma\)-ideal.

Proposition 6.3 immediately gives the following corollary.

Corollary 6.6. If \(X\) is \(\rho\)-deficient, then there is no \(\rho\)-invariant Borel probability measure.

This corollary is all we need about \(\rho\)-flows in our proofs below. However, it is well worth pointing out that its converse is also true (much more difficult to prove) and it is the content of [Mil08, Theorem 3]. We restate this here in our terms for the sake of completeness:

Theorem 6.7 (Miller 2008). For a countable Borel equivalence relation \(E\) on a standard Borel space \(X\) and a Borel cocycle \(\rho : E \to \mathbb{R}^+\),

(I) either: \(X\) is \(\rho\)-deficient,

(II) or else: there is a \(\rho\)-invariant Borel probability measure on \(X\).

6.D. Building a flow

Lemma 6.8. For any disjoint nonempty \(\rho\)-finite subsets \(U, V\) of the same \(E\)-class in \(X\) and a function \(w : U \to [0, +\infty)\), if \(\rho(V) \geq \int_U w \, d\rho\), then there is a \(\rho\)-flow \(\varphi\) with \(\text{dom}(\varphi) \subseteq U \times V\)
such that \( \text{out}^\rho \varphi|_U = w|_U \). In particular,

\[
\int_V \text{in}^\rho \varphi \, d\rho = \int_U w \, d\rho.
\]

**Proof.** The last equality is due to Lemma 6.2.

Recall that \( >_\rho \) is a Borel linear order on \( X \) whose restriction on \( \rho \)-finite sets is of order type \( \leq \omega \), so we may order \( U = \{u_n\}_{n<N}, V = \{v_m\}_{m<M} \), with \( N, M \leq \infty \), in the \( <_\rho \)-decreasing order. We inductively build a sequence \( (\varphi_n)_{n<N} \) of \( \rho \)-flows such that

(i) \( \text{dom}(\varphi_n) \subseteq \{u_n\} \times V \),
(ii) \( \overline{\varphi}_{n+1} := \sum_{k<n+1} \varphi_k \) is a \( \rho \)-flow,
(iii) \( \text{out}^\rho \varphi_n(u_n) = w(u_n) \).

 Granted this, taking \( \varphi := \overline{\varphi}_N \) clearly works.

Fixing \( n < N \), we suppose that the \( \varphi_{n'} \) are defined as desired for all \( n' < n \). We define the value \( \varphi_n(u_n, v_m) \) by induction on \( m \) as the maximum of the quantities

\[
(1 - \text{in}^\rho \overline{\varphi}_n(v_m)) \frac{\rho(v_m)}{\rho(u_n)} \quad \text{and} \quad w(u_n) - \sum_{m' < m} \varphi_n(u_n, v_{m'}),
\]

so

\[
\text{in}^\rho \overline{\varphi}_n(v_m) + \varphi_n(u_n, v_m) \cdot \frac{\rho(u_n)}{\rho(v_m)} \leq 1 \quad \text{and} \quad \sum_{m' \leq m} \varphi_n(u_n, v_{m'}) \leq w(u_n),
\]

and the equality holds in at least one of these inequalities. This finishes the construction of \( \varphi_n \) and condition \( \rho(V) \geq \int_U w \, d\rho \) and the induction hypothesis guarantee that \( \text{out}^\rho \varphi_n(u_n) = w(u_n) \).

The proof of the last lemma can be carried out in a uniformly Borel fashion, yielding:

**Lemma 6.9.** Let \( F \subseteq E \) be a \( \rho \)-finite Borel subequivalence relation\(^{15} \), \( U, V \subseteq X \) be disjoint Borel sets, and \( w : U \to [0, \infty) \) a Borel function. If for each \( F \)-class \( Y \), \( \rho(V \cap Y) \geq \int_{U \cap Y} w \, d\rho \), then there exists a Borel \( \rho \)-flow \( \varphi \) with \( \text{dom}(\varphi) \subseteq (U \times V) \cap F \) and \( \varphi|_U = w|_U \). In particular, for each \( F \)-class \( Y \),

\[
\int_{V \cap Y} \text{in}^\rho \varphi \, d\rho = \int_{U \cap Y} w \, d\rho.
\]

### 7. Packing and saturation

Throughout this section, let \( X \) be a standard Borel space, \( E \) be a countable Borel equivalence relation on \( X \), and \( \rho : E \to \mathbb{R}^+ \) a Borel cocycle. Within a given Borel collection \( \mathcal{F} \) of \( \rho \)-finite subsets of \( X \), we will build Borel *prepartitions* of various degrees of maximality.

In the pmp setting, i.e. when \( \rho \equiv 1 \), all of the results below were proven in [MT17, Section 4].

\(^{15}\)We call an equivalence relation \( F \subseteq E \) *\( \rho \)-finite* if each \( F \)-class is \( \rho \)-finite.
7.A. Prepartitions

For any $\mathcal{F} \subseteq [X]_{E}^{\rho<\infty}$, call the set

$$\text{dom}(\mathcal{F}) := \{ x \in X : \exists A \in \mathcal{F} \text{ with } x \in A \}$$

the **domain** of $\mathcal{F}$. When $\mathcal{F}$ is Borel, this set is analytic (hence measurable) in general, but it is Borel in all of the situations we deal with below, see Lemma 7.2. Also, for $Y \subseteq X$, put

$$\mathcal{F}|_Y := \{ U \in \mathcal{F} : U \subseteq Y \}.$$ 

**Definition 7.1.** Let $\mathcal{F}, \mathcal{S} \subseteq [X]_{E}^{\rho<\infty}$. We say that $\mathcal{F}$ is

- a **prepartition** (within $\mathcal{S}$) if the sets in $\mathcal{P}$ are pairwise disjoint (and belong to $\mathcal{S}$). In this case, denote by $E(\mathcal{F})$ the equivalence relation that is the identity outside of $\text{dom}(\mathcal{F})$ and on $\text{dom}(\mathcal{F})$ its classes are exactly the elements of $\mathcal{F}$.
- **finitely based** if for any $A \in \mathcal{F}$ and any finite $B \subseteq A$ there is a finite $B' \in \mathcal{F}$ with $B \subseteq B' \subseteq A$.
- **$\rho$-approximable** if for any $A \in \mathcal{F}$ there is $\epsilon \in (0, 1)$ such that all $B \subseteq A$ with $\rho(B) \geq (1-\epsilon) \cdot \rho(A)$ belong to $\mathcal{F}$. In particular, such $\mathcal{F}$ is finitely based.
- **upward continuous** if for every increasing sequence $(A_n) \subseteq \mathcal{F}$, if $\bigcup_{n} A_n$ is $\rho$-finite, then it belongs to $\mathcal{F}$.

**Lemma 7.2.** If a Borel collection $\mathcal{F} \subseteq [X]_{E}^{\rho<\infty}$ is either finitely based or a prepartition, then $\text{dom}(\mathcal{F})$ is Borel.

**Proof.** In the first case, $\text{dom}(\mathcal{F}) = \text{dom}(\mathcal{F} \cap [X]_{E}^{\rho<\infty})$ and every $x \in \text{dom}(\mathcal{F} \cap [X]_{E}^{\rho<\infty})$ is contained in only countably-many sets in $\text{dom}(\mathcal{F} \cap [X]_{E}^{\rho<\infty})$, so the Luzin–Novikov theorem implies that $\text{dom}(\mathcal{F})$ is Borel. In the second case, every $x \in \text{dom}(\mathcal{F})$ is contained in exactly one set in $\mathcal{F}$, so $\text{dom}(\mathcal{F})$ is Borel by the Luzin–Souslin theorem. 

**Definition 7.3.** Let $\mathcal{P}, \mathcal{P}' \subseteq [X]_{E}^{\rho<\infty}$ be prepartitions. We say that $\mathcal{P}'$ is a **partial extension** of $\mathcal{P}$, noted $\mathcal{P}' \triangleright \mathcal{P}$, if each set in $\mathcal{P}'$ is $E(\mathcal{P})$-invariant. If, moreover, $\text{dom}(\mathcal{P}') \supseteq \text{dom}(\mathcal{P})$, we say that $\mathcal{P}'$ is an extension of $\mathcal{P}$ and write $\mathcal{P}' \triangleright \mathcal{P}$.

Lastly, we call a sequence $(\mathcal{P}_n)$ of prepartitions **coherent** if $\mathcal{P}_i \leq \mathcal{P}_j$ for all $i \leq j$. In this case, observe that each $\bigcup_{n \leq N} E(\mathcal{P}_n)$ is an equivalence relation for each $N \leq \infty$. Thus it makes sense to define

$$\lim_{n \to \infty} \mathcal{P}_n$$

as the collection of all equivalence classes of the relation $\bigcup_{n \in \mathbb{N}} E(\mathcal{P}_n)$ that are contained in $\bigcup_{n \in \mathbb{N}} \text{dom}(\mathcal{P}_n)$. Lastly, we say that the sequence $(\mathcal{P}_n)$ stabilizes if $\lim_{n \to \infty} \mathcal{P}_n \subseteq \bigcup_n \mathcal{P}_n$.

7.B. Packed prepartitions

**Definition 7.4.** Let $p \in \mathbb{R}^+$. 

- For a prepartition $\mathcal{P} \subseteq [X]_{E}^{\rho<\infty}$, call a set $A \subseteq [X]_{E}^{\rho<\infty}$ a **$p$-pack** over $\mathcal{P}$ if $A$ is $E(\mathcal{P})$-invariant and $\rho(A \setminus \text{dom}(\mathcal{P})) \geq p \cdot \rho(A \cap \text{dom}(\mathcal{P}))$, equivalently, $\rho(A) \geq (1+p) \cdot \rho(A \cap \text{dom}(\mathcal{P}))$.
- For a collection $\mathcal{F} \subseteq [X]_{E}^{\rho<\infty}$, call a prepartition $\mathcal{P} \subseteq \mathcal{F}$ **$p$-packed within $\mathcal{F}$** if $\mathcal{F}$ has no $p$-pack over $\mathcal{P}$.
• A *p-packing sequence* is an extension-increasing sequence \((\mathcal{F}_n)\) of prepartitions contained in \([X]_{E}^{p<\infty}\) such that for each \(n\), each set in \(\mathcal{F}_{n+1} \setminus \mathcal{F}_n\) is a \(p\)-pack over \(\mathcal{F}_n\).

We drop \(p\) from the notation and terminology if it is equal to 1.

**Observation 7.5.** For any \(p \in \mathbb{R}^+\), any prepartition \(\mathcal{P}\) \(p\)-packed within a collection \(\mathcal{F} \subseteq [X]_{E}^{p<\infty}\) is, in particular, maximal within \(\mathcal{F}\), i.e. there is no \(U \in \mathcal{F}\) disjoint from \(\text{dom}(\mathcal{P})\).

**Terminology 7.6.** We say that a statement holds **modulo \(p\)-deficient** if it holds on \(X \setminus D\) for some \(E_G\)-invariant \(p\)-deficient Borel set.

**Lemma 7.7.** Any \(p\)-packing sequence \((\mathcal{P}_n)\) of Borel prepartitions within \([X]_{E}^{p<\infty}\) stabilizes modulo \(p\)-deficient.

**Proof.** By replacing \(p\) with \(\min\{1, p\}\), we assume, without loss of generality, that \(p \in (0, 1]\). Let \(Z\) be the union of all \(E\)-classes \(C\) such that \((\mathcal{P}_n|_C)\) does not stabilize and we assume, without of generality that \(X = Z\).

Put \(E_n := \text{E}(\mathcal{P}_n)\) and \(D_n := \text{dom}(\mathcal{P}_k)\) for each \(n \in \mathbb{N}\). We will recursively define a sequence \((\varphi_n)\) of Borel \(p\)-flows with pairwise disjoint domains, whose sum \(\varphi_\infty := \sum_n \varphi_n\) is a \(p\)-flow which disbalances \(X\), thus finishing the proof. For each \(n \in \mathbb{N}\), putting \(\varphi_n := \sum_{k<n} \varphi_k\), where \(\varphi_0\) is, by definition, the zero \(p\)-flow, we require that

\[
\begin{align*}
(7.7.i) & \quad \text{dom}(\varphi_n) \subseteq E_{n+1} \cap D_n \times (D_{n+1} \setminus D_n); \\
(7.7.ii) & \quad \text{all the sources and sinks of } \varphi_n \text{ are pure, i.e. } \text{in}^p \varphi_n(x) = 0 \text{ for a source } x \text{ and } \text{out}^p \varphi_n(x) = 0 \text{ for a sink } x; \\
(7.7.iii) & \quad \text{for each } E_n\text{-class } Y, \text{Sinks}(\varphi_n) \cap Y = \emptyset \Rightarrow \text{vdom}(\varphi_n) \cap Y = \emptyset; \\
(7.7.iv) & \quad \text{for each } E_n\text{-class } Y, \int_{\text{Sinks}(\varphi_n) \cap Y} \text{in}^p \varphi_n d\rho \leq p \cdot \rho(Y). 
\end{align*}
\]

Turning to the construction, for each \(n \in \mathbb{N}\), put

\[
\begin{align*}
X_n & := \left\{ x \in D_{n+1} : [x]_{E_{n+1}} \cap D_n, [x]_{E_{n+1}} \setminus D_n \neq \emptyset \right\} \\
U_n & := X_n \cap D_n \\
V_n & := X_n \setminus D_n
\end{align*}
\]

and observe that \(X_n\) is \(E_{n+1}\)-invariant while \(U_n\) is \(E_n\)-invariant. Now fix \(n \in \mathbb{N}\) and suppose that \(\varphi_0, \ldots, \varphi_{n-1}\) are defined and satisfy (7.7.i)–(7.7.iv). We will define the \(p\)-flow \(\varphi_n\) only on \(E_{n+1} \cap (U_n \times V_n)\) using Lemma 6.9. First, we define \(w_n : U_n \to [0, 1]\) as follows: for each \(E_{n+1}\)-class \(Y \subseteq X_n\), \(w_n|_{U_n \cap Y} := 1\) if \(\varphi_n\) has no sinks in \(U_n \cap Y\); otherwise, for \(y \in U_n \cap Y\),

\[w_n(y) := \mathbb{1}_{\text{Sinks}(\varphi_n)}(y) \cdot \text{in}^p \varphi_n(y).
\]

Note that in the latter case, using (7.7.iv),

\[
\int_{U_n \cap Y} w_n d\rho = \int_{\text{Sinks}(\varphi_n) \cap Y} \text{in}^p \varphi_n d\rho \leq p \cdot \rho(U_n \cap Y).
\]

Thus, in either case,

\[
\int_{U_n \cap Y} w_n d\rho \leq p \cdot \rho(U_n \cap Y). \tag{7.8}
\]

Because \(V_n \cap Y\) is a \(p\)-pack over \(\mathcal{P}_n\),

\[
\rho(V_n \cap Y) \geq p \cdot \rho(U_n \cap Y) \Rightarrow \int_{U_n \cap Y} w_n d\rho,
\]

24
so Lemma 6.9 applies to \( E_{n+1}, U_n, V_n, w_n \), and gives the desired \( p \)-flow \( \varphi_{n+1} \).

By definition, (7.7.i) holds for \( \varphi_n \). Furthermore, if \( \varphi_n \) has no sinks in \( U_n \cap Y \), then, by (7.7.iii), it has no sources either, so Sources\( (\varphi_{n+1}) = U_n \cap Y \) and Sinks\( (\varphi_{n+1}) \cap Y \subseteq V_n \) by the definition of \( \varphi_n \) in this case, and all the sources and sinks are pure. In the other case, no new sources are introduced because \( \partial^o \varphi_{n+1}|_{U_n \cap Y} = 0 \) and we again have that Sinks\( (\varphi_{n+1}) \cap Y \subseteq V_n \), so all sinks of \( \varphi_{n+1} \) are pure and, due to Lemma 6.9 and (7.8),

\[
\int_{\text{Sinks}(\varphi_{n+1}) \cap Y} \inf^o \varphi_{n+1} \, d\rho = \int_{U_n \cap Y} \inf^o \varphi_{n+1} \, d\rho = \int_{U_n \cap Y} w_n \, d\rho \leq p \cdot \rho(U_n \cap Y) \leq p \cdot \rho(Y),
\]

verifying (7.7.iv) for \( \varphi_{n+1} \). This completes the construction of the sequence \( (\varphi_n) \).

Let \( S \) be the set of all \( x \in X \) such that \( x \in U_{n(x)} \), where \( n(x) \) is the least \( k \in \mathbb{N} \) with \( X_k \cap [x]_E \neq \emptyset \).

**Claim.** \( S \subseteq \text{Sources}(\varphi_\infty) \).

**Proof of Claim.** It is clear that from the definitions that each \( x \in S \) is a \( \varphi_{n(x)+1} \)-source, so it remains to note that if a point \( x \in X \) is a \( \varphi_k \)-source for some \( k \geq 0 \), then \( x \notin \text{vdom}(\varphi_m) \) for all \( m > k \), so its in-flow and out-flow remain unchanged through the summation, and hence, it is a \( \varphi_\infty \)-source.

It remains to argue that \( \varphi_\infty \) does not have sinks. Indeed, each point \( x \in \text{vdom}(\varphi_\infty) \) that is not a \( \varphi_\infty \)-source appears in the vertex-domain of a \( \varphi_n \) exactly twice: once in a \( V_n \), where it is turned into a pure \( \varphi_n \)-sink, and another time in a \( U_m \), for some \( m > n \), where it is turned into a pure \( \varphi_m \)-source with out\( \partial^o \varphi_m(x) = \inf^o \varphi_m(x) = \inf^o \varphi_{n+1}(x) \). Thus, for any \( k \geq m + 1 \), \( \partial^o \varphi_k(x) = 0 \), so \( \partial^o \varphi_\infty(x) = 0 \). \( \square \)

**Theorem 7.9.** For any countable Borel equivalence relation \( E \) on \( X \), \( p \in \mathbb{R}^+ \), and a Borel \( F \subseteq [X]^{<\infty} \), there is a Borel prepartition \( \mathcal{P} \subseteq F \) that is \( p \)-packed within \( F \) modulo \( p \)-deficient.

**Proof.** By [KM04, Proof of Lemma 7.3], the intersection graph on \( [X]^{<\infty} \) admits a countable Borel coloring and we fix one. Let \( (k_n) \) be a sequence of natural numbers in which each \( k_n \in \mathbb{N} \) appears infinitely-many times.

We recursively build a \( p \)-packing sequence \( (\mathcal{P}_n) \), with \( \mathcal{P}_n \subseteq F \) being a Borel subset, as follows. Take \( \mathcal{P}_0 := \emptyset \) and, fixing \( n \geq 1 \), suppose that \( \mathcal{P}_{n-1} \) is defined. Let \( \mathcal{P}_n \) be the collection of all sets in \( F \) of color \( k_n \) that are \( p \)-packs over \( \mathcal{P}_{n-1} \) and let \( \mathcal{P}_n := \mathcal{P}_n \cup \mathcal{P}_{n-1}|_{X \setminus \text{dom}(\mathcal{P}_n)} \).

By Lemma 7.7, we may assume that the sequence \( (\mathcal{P}_n) \) stabilizes, so \( \mathcal{P} := \lim_n \mathcal{P}_n \subseteq F \) and it remains to show that \( \mathcal{P} \) is \( p \)-packed within \( F \).

Suppose towards a contradiction that \( U \in F \) is a \( p \)-pack over \( \mathcal{P} \). Let \( N \) be large enough so that \( \mathcal{P}|_U = \mathcal{P}_N|_U \) and hence, \( U \) is a \( p \)-pack over \( \mathcal{P}_n \) for all \( n \geq N \). Letting \( k \) be the color of \( U \), there are arbitrarily large \( n \) with \( k_n = k \), so there must be \( n > N \) for which \( U \) is in \( \mathcal{P}_n \), a contradiction. \( \square \)

**Lemma 7.10.** Let \( F \subseteq [X]^{<\infty}_E \) be a Borel approximable collection, \( p \in \mathbb{R}^+ \), and \( \mathcal{P} \subseteq F \cap [X]^{<\infty}_E \) a Borel prepartition \( p \)-packed within \( F \cap [X]^{<\infty}_E \). Then, for any \( p' > p \), \( \mathcal{P} \) is \( p' \)-packed within \( F \); in fact, any prepartition \( \mathcal{P}' \subseteq F \) extending \( \mathcal{P} \) is \( p' \)-packed.

**Proof.** Let \( p' > p \) and let \( \mathcal{P}' \subseteq F \) be extending \( \mathcal{P} \). Suppose towards a contradiction that there is a \( p \)-pack \( A \) over \( \mathcal{P}' \) and let \( \epsilon \in (0, 1) \) be as in Definition 7.1 of approximability for \( A \). Taking a positive \( \delta \leq \epsilon \) such that \((1 - \delta)(p' + 1) \geq (p + 1)\), there is an \( E(\mathcal{P}) \)-invariant finite
subset $B \subseteq A$ with $\rho(B) \geq (1 - \delta) \cdot \rho(A)$. But then
\[
\rho(B) \geq (1 - \delta) \cdot (1 - \delta) \cdot \rho(A) \geq (1 - \delta)(p' + 1) \cdot \rho(A \cap \text{dom}(P)) \geq (p + 1) \cdot \rho(B \cap \text{dom}(P)),
\]
contradicting the $p$-packedness of $P$ within $F \cap [X]_E^\infty$. \hfill \Box

### 7.C. Saturated prepartitions

**Definition 7.11.** Let $p \in \mathbb{R}^+$.

- For a prepartition $P \subseteq [X]_E^{\rho < \infty}$, call a set $A \in [X]_E^{\rho < \infty}$ injective over $P$ if $A$ is $E(P)$-invariant and contains at most one set from $P$.
- For a collection $F \subseteq [X]_E^{\rho < \infty}$, call a prepartition $P \subseteq F$ $p$-saturated within $F$ if there is no injective $p$-pack over $P$ in $F$. Call $P$ saturated within $F$ if it is $p$-saturated within $F$ for every $p > 0$, equivalently, if there is no injective set over $P$ in $F \setminus P$.
- Call an extension-increasing sequence $(F_n)$ of prepartitions contained in $[X]_E^{\rho < \infty}$ injective if for each $n$, each set in $F_{n+1}$ is injective over $F_n$.

**Proposition 7.12.** Let $F \subseteq [X]_E^{\rho < \infty}$ be a Borel collection and $S_0 \subseteq F$ be a Borel prepartition. For any $p \in \mathbb{R}^+$, there is a Borel collection $S_0 \subseteq S \subseteq F$ that is $p$-saturated within $F$ modulo $\rho$-deficient.

**Proof.** The proof is the same as that of Theorem 7.9 with every appearance of “$p$-pack” replaced by “injective $p$-pack” and $\rho_0$ taken to be $S_0$. \hfill \Box

**Lemma 7.13.** For an injective sequence $(P_n)$ of Borel collections $P_n \subseteq [X]_E^{\rho < \infty}$, the $\rho$-aperiodic part $Z \subseteq X$ of $E_\infty := \bigcup_n E(P_n)$ is $\rho$-deficient. In particular, $[Z]_E$ is $\mu$-null for any $\rho$-invariant Borel probability measure $\mu$ on $X$.

**Proof.** Let $E_n := E(P_n)$ for each $n$ and let $U_0$ be the union of inclusion-minimal sets in $\bigcup_n P_n$; in particular, $[U_0]_{E_\infty} = Z$.

We define $k_m : U_0 \rightarrow \mathbb{N}$ by induction on $m$ as follows. For each $x \in U_0$, let $k_0(x)$ be the least $k \in \mathbb{N}$ with $[x]_{E_k} \subseteq U_0 \cap \text{dom}(P_k)$, so $[x]_{E_k} \in P_k$. Supposing that $k_m(x)$ is defined, let $k_{m+1}(x)$ be the least $k > k_m(x)$ such that
\[
\rho([x]_{E_k \setminus [x]_{E_{k_m(x)}}}) \geq \rho([x]_{E_{k_m(x)}}).
\]
Observe that for each $m \in \mathbb{N}$, due to the injectivity of the sequence $(P_n)$
\[
F_m := \{[x]_{E_{k_m(x)}} : x \in U_0\}
\]
is a prepartition, by the , and let $F_m := E(F_m)$. For each $m \in \mathbb{N}$, let
\[
U_{m+1} := \bigcup \{[x]_{F_{m+1} \setminus [x]_{F_m}} : x \in S\},
\]
so $\{U_m\}_{m \in \mathbb{N}}$ is a partition of $Z$ into Borel sets, for each $m \in \mathbb{N}$, the set $\bigcup_{i \leq m} U_i$ is $F_m$-invariant and for each $Y \in F_m$, $\rho(U_m \cap Y) \geq \rho(U_0 \cap Y)$.

It is clear now that recursive applications of Lemma 6.9 with $U := U_m$ and $V := U_{m+1}$ would yield a sequence $(\varphi_m)$ of Borel $\rho$-flows such that
\[
\text{dom}(\varphi_m) \subseteq (U_m \times U_{m+1}) \cap F_{m+1}
\]
(7.14)

and for each $Y \in \mathcal{F}_{m+1}$,

$$\int_{U_m \cap Y} \text{out}^p \varphi_m \, d\rho = \int_{U_{m+1} \cap Y} \text{in}^p \varphi_m \, d\rho = \rho(U_0 \cap Y). \quad (7.15)$$

Letting $\varphi$ be the sum $\sum_m \varphi_m$, we observe that $\varphi$ is 1-bounded, $\text{Sources}(\varphi) = U_0$, but there are no $\varphi$-sinks, so $\varphi$ creates a deficit throughout $Z$. \hfill \square

7.D. Packed and saturated prepartitions

**Theorem 7.16.** For any $p \in \mathbb{R}^+$, any Borel approximable upward continuous collection $\mathcal{F} \subseteq [X]^{p<\infty}_E$ admits a Borel collection $\mathcal{P} \subseteq \mathcal{F}$ that is both saturated and $p$-packed within $\mathcal{F}$ modulo $\rho$-deficient.

**Proof.** Put $\mathcal{F}' := \mathcal{F} \cap [X]^{\infty}_E$ and take a decreasing sequence $(p_n)$ of positive reals converging to 0 with $p_0 := \frac{p}{2}$. In the course of the proof, we will throw out countably-many $\rho$-deficient sets without mentioning.

Theorem 7.9 gives a Borel collection $\mathcal{P}_0 \subseteq \mathcal{F}'$ that is $p_0$-packed within $\mathcal{F}'$. Iterative applications of Proposition 7.12 give an injective sequence $(\mathcal{P}_n)$ of Borel prepartitions contained in $\mathcal{F}'$ such that $\mathcal{P}_n$ is $p_n$-saturated within $\mathcal{F}'$. Put $\mathcal{P} := \lim_n \mathcal{P}_n$; Lemma 7.13 implies that $\mathcal{P} \subseteq [X]^{p<\infty}_E$ modulo a $\rho$-deficient set, which we ignore. It then follows by the upward continuity of $\mathcal{F}$ that $\mathcal{P} \subseteq \mathcal{F}$.

Lemma 7.10 implies that $\mathcal{P}$ is $p$-packed within $\mathcal{F}$, so it remains to check that $\mathcal{P}$ is also saturated within $\mathcal{F}$. Towards a contradiction, let $A \in \mathcal{F} \setminus \mathcal{P}$ be injective over $\mathcal{P}$ and put $P := A \cap \text{dom}(\mathcal{P})$. Let $\varepsilon \in (0,1)$ be as in Definition 7.1 of approximability for $A$ and take a positive $\delta \leq \varepsilon$ such that $(1 - \delta) \cdot \rho(A) \geq (1 + \delta) \cdot \rho(P)$. Letting $n \in \mathbb{N}$ be such that $p_n < \delta$ and $P_n := A \cap \text{dom}(\mathcal{P}_n)$, we take any $A' \in [X]^{\infty}_E$ such that $P_n \subseteq A' \subseteq A$ and $\rho(A') \geq (1 - \delta) \cdot \rho(A)$; in particular, $A' \in \mathcal{F}'$. It then follows that $A' \cap \text{dom}(\mathcal{P}_n) = P_n$ and $\rho(A') > (1 + p_n) \cdot \rho(P_n)$, so $A'$ is an injective $p_n$-pack over $\mathcal{P}_n$, contradicting the $p_n$-packedness of $\mathcal{P}_n$ within $\mathcal{F}$.

Lastly, we state a version of Theorem 7.16 relative to a $\rho$-finite Borel equivalence relation $F$. For a collection $\mathcal{S} \subseteq [X]^{p<\infty}_E$ of $F$-invariant sets, let $\mathcal{S}_F$ denote its natural image under the quotient map $X \to X/F$.

**Corollary 7.17.** For any $p \in \mathbb{R}^+$, any $\rho$-finite Borel subequivalence relation $F \subseteq E$, and any Borel upward continuous collection $\mathcal{S} \subseteq [X]^{p<\infty}_E$ of $F$-invariant sets such that $\mathcal{S}_F$ is $\rho_F$-approximable, there is a Borel collection $\mathcal{P} \subseteq \mathcal{S}$ that is both saturated and $p$-packed within $\mathcal{S}$ modulo $\rho$-deficient.

**Proof.** We apply Theorem 7.16 to the quotients by $F$ of all of the objects involved and pull back the resulting saturated and $p$-packed collection. Using the smoothness of $F$, it is easy see that the pull-back of a $\rho_F$-deficient set is $\rho$-deficient because a $\rho_F$-flow with no sinks lifts to a $\rho$-flow with no sinks. \hfill \square

### 8. Cocycled graph visibility

Throughout this section, let $G$ be a locally countable Borel graph on a standard Borel space $X$ and let $\rho : E_G \to \mathbb{R}^+$ be a Borel cocycle.
8.A. Definitions and basic properties

In our proof, we would like to obtain $\rho$-finite $G$-connected sets of arbitrarily large $\rho$-ratio. In other words, we would like to obtain $\rho$-large $G$-connected sets of points whose individual mass is bounded. This is capture in the following definition.

**Definition 8.1.** Let $x \in X$ and $\alpha \in [1, \infty)$.

- A \((G, \rho, \alpha)\)-visible neighborhood of $x$ is any $G$-connected set $V \ni x$ such that $\alpha \times \rho(x) \geq \rho(v)$ for each $v \in V$.
- A point $y \in X$ is said to be $(G, \rho, \alpha)$-visible from $x$ if $x$ admits a $(G, \rho, \alpha)$-visible neighborhood containing $y$.
- We call the set $B^G_p(x, \alpha)$ of all points in $X$ $(G, \rho, \alpha)$-visible from $x$ the $(G, \rho, \alpha)$-block of $x$. It is immediate that $B^G_p(x, \alpha)$ is $G$-connected.
- Call a set $B$ a $(G, \rho, \alpha)$-block if $B = B^G_p(x, \alpha)$ for some $x \in X$; call any such $x$ an $\alpha$-dominus of $B$. We call $B$ a bounded $(G, \rho)$-block if $B = B^G_p(x, \alpha)$ for some $x \in X$ and $\alpha \in [1, \infty)$. We say that $B$ is proper if it is not equal to an entire $G$-connected component.
- We call the quantity $v^G_p(x, \alpha) := \frac{1}{\alpha} \rho^\chi(B^G_p(x, \alpha))$ the $(G, \rho, \alpha)$-visibility at $x$.
- We say that $G$ has finite $\rho$-visibility if for each $x \in X$, $v^G_p(x)$ is finite; equivalently, $v^G_p(x, \alpha)$ is finite for each $\alpha \in [1, \infty)$.

We omit $G, \rho$ from notation if they are understood from the context and we omit $\alpha$ if it is 1; for instance, by an “$\alpha$-block” we mean a $(G, \rho, \alpha)$-block and by a “block” a 1-block.

**Proposition 8.2.** Let $G, \rho$ be as above.

**8.2.a** (Nested or disjoint) Any two bounded blocks are either nested (i.e. one is contained in the other) or disjoint.

**8.2.b** (Covering $\rho$-finite) Every $U \in [X]_{G, \rho}^{\rho < \infty}$ is contained in a bounded block.

**8.2.c** (Amalgamation) Any two bounded blocks $B^G_p(x, \alpha)$ and $B^G_p(y, \beta)$ in the same $G$-connected component are contained in a single max $[\alpha, \beta]$-block.

**Proof.** *(8.2.a):* For bounded blocks $B^G_p(x, \alpha)$ and $B^G_p(y, \beta)$ with $\alpha \cdot \rho(x) \geq \beta \cdot \rho(y)$, if they intersect then $B^G_p(x, \alpha) \supseteq B^G_p(y, \beta)$.

*(8.2.b):* Taking any $x \in U$ and $\alpha := \max\{\rho^\chi(U), 1\}$, $B^G_p(x, \alpha) \supseteq U$.

*(8.2.c):* For bounded blocks $B^G_p(x, \alpha)$ and $B^G_p(y, \beta)$, letting $P$ be a $G$-path from $x$ to $y$, we see that $B^G_p(\max_{< \rho} P, \max [\alpha, \beta]) \supseteq B^G_p(x, \alpha) \cup P \cup B^G_p(y, \beta)$. \(\square\)

**Notation 8.3.** For a subset $A$ of an $E_G$-class, put

$$\min_{\rho} A := \{x \in A : \forall y \in A \rho(x) \leq \rho(y)\}.$$  

In $\rho$-homogeneous expressions, when $\min_{\rho} A = \emptyset$, we write $\min_{\rho} A$ to mean $\rho(x)$ for some (equivalently, any) $x \in \min_{\rho} A$. We also analogously define $\max_{\rho} A$ and $\max_{\rho} A$.

8.B. Finite visibility and hyperfiniteness

**Theorem 8.4.** Let $G$ be a locally countable Borel graph on a standard Borel space $X$ and let $\rho : E_G \to \mathbb{R}^+$ be a Borel cocycle. If $G$ has finite $\rho$-visibility, then $E_G$ is hyperfinite.
The rest of this subsection is devoted to the proof of Theorem 8.4, so we suppose that \(G\) has finite \(\rho\)-visibility.

For each \(x \in X\), we consider its \((G, \rho)\)-cone, namely, the set
\[
C^G_\rho(x) := \left\{ y \in [x]_E : x \in B^G_\rho(y) \right\}
\]
and its \(\rho^x\)-image
\[
c^G_\rho(x) := \left\{ \rho^x(y) : y \in C^G_\rho(x) \right\},
\]
and we use the following lemma below without mention.

**Lemma 8.5.** Let \(x \in X\).
1. (8.5.a) For each \(y \in X\), \(y \in C^G_\rho(x) \Rightarrow C^G_\rho(y) \subseteq C^G_\rho(x)\).
2. (8.5.b) \(c^G_\rho(x)\) is cofinal in \(\left\{ \rho^x(y) : y \in [x]_E \right\}\), i.e. \(\sup c^G_\rho(x) = \sup \left\{ \rho^x(y) : y \in [x]_E \right\}\).

**Proof.** (8.5.a) is immediate. As for (8.5.b), let \(P\) be a \(G\)-path from \(x\) to \(y\) and let \(z \in \text{Max}_\rho P\). Then \(B^G_\rho(z) \supseteq \{x, y\}\), so \(y \in c^G_\rho(x)\) and \(\rho(z) \geq \rho(y)\).

**Claim 8.6.** For each \(x \in X\), \(c^G_\rho(x)\) has at most one limit point in \((0, \infty]\), namely, its supremum.

**Proof.** Indeed, if \(\ell \in (0, \infty]\) is a limit point different from \(\sup c^G_\rho(x)\), then there is \(y \in [x]_E\) with \(x \in B^G_\rho(y)\) and \(\rho^x(y) > \ell\). But then \(B^G_\rho(y)\) contains every \(z \in c^G_\rho(x)\) with \(\rho(x) \leq \rho(z) < \rho(y)\) and there are infinitely-many of them since \(\ell\) is a limit point. This implies that \(B^G_\rho(y)\) is \(\rho\)-infinite, contradicting finite \(\rho\)-visibility. \(\square\)

**Definition 8.7.** For blocks \(B, C\), we say that \(C\) is the next block after \(B\) if it is the inclusion-least block properly containing \(B\). Such a block is unique by definition and we denote it by \(B'\), if it exists; otherwise \(B' := B\).

**Claim 8.8.** For any block \(B \subseteq X\) and any point \(y \in \partial^\text{out}_G B\), the set
\[
(\partial^\text{out}_G B)_{\leq \rho(y)} := \left\{ z \in \partial^\text{out}_G B : \rho(z) \leq \rho(y) \right\}
\]
is finite. In particular, \(\text{Min}_\rho \partial^\text{out}_G B\) is nonempty.

**Proof.** Each point \(z \in (\partial^\text{out}_G B)_{\leq \rho(y)}\) is \((G, \rho)\)-visible from \(y\), so \((\partial^\text{out}_G B)_{\leq \rho(y)} \subseteq B^G_\rho(y)\), and hence, \((\partial^\text{out}_G B)_{\leq \rho(y)}\) is \(\rho\)-finite, by finite \(\rho\)-visibility. On the other hand, letting \(x \in X\) be a dominus of \(B\), each point \(z \in (\partial^\text{out}_G B)_{\leq \rho(y)}\) is \(\rho\)-greater than \(x\), so there can be only finitely many such \(z\). \(\square\)

**Claim 8.9.** Every proper block \(B\) admits a next block, namely, \(B^G_\rho(y)\) for some (equivalently, any) \(z \in \text{Min}_\rho \partial^\text{out}_G B\).

**Proof.** This follows from the fact that any block \(C \supset B\) has to contain at least one point from \(y \in \partial^\text{out}_G B\), so \(C \supset B^G_\rho(y) \supset \text{Min}_\rho \partial^\text{out}_G B\). But \(\text{Min}_\rho \partial^\text{out}_G B \not= \emptyset\) by Claim 8.8. \(\square\)

We fix a Borel selector \(s : [X]^{<\infty} \to X\) and, in light of Claim 8.9, define \(f : X \to X\) by \(x \mapsto s(\text{Max}_\rho B^G_\rho(x'))\).

**Observation 8.10.** For every \(x \in X\),
1. (8.10.a) \(f(x) \in C^G_\rho(x)\).
As before, throughout this subsection, we let $G$ of visible neighborhood.

Lemma 8.14. Let $\rho : E_G \to \mathbb{R}^+$ be a Borel cocycle. Moreover, we let $\mu$ be a $\rho$-invariant Borel probability measure on $X$.

The goal of this subsection is to build a Borel prepartition $P$ with large domain whose cells all have a large $\rho$-ratio. This is not hard in the pmp setting, i.e. when $\rho$-ratio is just size; indeed, just taking a saturated prepartition works. However, when $\rho$-ratio is not just size and is moreover not monotone, such a prepartition is harder build.

Firstly, we extend the definition of a visible neighborhood from points to sets: a $(G,\rho)$-visible neighborhood of $U \in [X]_G^{\rho<\infty}$ is any $G$-connected set $V \supseteq U$ such that $\rho(U) \geq \rho(v)$ for each $v \in V$.

For a collection $S \subseteq [X]_G^{\rho<\infty}$, a function $\varphi : S \to [0,\infty)$, and $L \in [0,\infty)$, let $S[\varphi \geq L]$ denote the subcollection of all sets $U \in S$ with $\varphi(U) \geq L$.

Lemma 8.14. Let $X, G, \rho, \mu$ be as above and let $S \subseteq [X]_G^{\rho<\infty}$ be a Borel collection of sets such that for each $U \in [X]_G^{\rho<\infty}$, all of its $\rho$-finite visible neighborhoods of large enough $\rho$-ratio are in $S$. If $G$ is nowhere $\mu$-hyperfinite\footnote{This means that there is no $E_G$-invariant Borel set of positive $\mu$-measure on which $G$ is hyperfinite.}, then for every $\varepsilon, L > 0$, there is a Borel prepartition $P \subseteq S[\rho^{\max} \geq L]$ with a $\mu$-co-$\varepsilon$ domain.

The rest of this subsection is devoted to the proof of Lemma 8.14, so we assume the hypothesis of the theorem and fix $\varepsilon, L > 0$. Let $(L_n)$ be an increasing unbounded sequence of positive reals with $L_0 \geq L$.

Terminology 8.15. Let $F$ be a $\rho$-finite $G$-connected equivalence relation.
Claim 8.17. Similarly, we let Max_{\rho/F} U and max_{\rho/F} U stand for Max_{\rho/F} (U/F) and max_{\rho/F} (U/F).

Lastly, for F-invariant U ∈ [X]^{0,∞}_G, call V ⊇ U a (G/F, \rho/F)-visible neighborhood of U if V is F-invariant and V/F is a (G/F, \rho/F)-visible neighborhood of U/F.

For any \rho-finite G-connected equivalence relation F ⊆ E_G and M > 0, let \mathcal{S}[F]\left[\rho/F\max \uparrow M\right] denote the collection of all F-invariant P ∈ \mathcal{S} with \rho(F\max(P)) ≥ M such that for some/any U ∈ Max_{\rho/F} P, all of the F-invariant (G/F, \rho/F)-visible neighborhoods V of U containing P are in \mathcal{S}.

Ignoring countably-many \rho-deficient sets, iterative applications of Corollary 7.17 give a coherent sequence (\mathcal{P}_n) of Borel prepartitions such that \mathcal{P}_n is saturated within

\mathcal{S}[F_{n-1}]\left[\rho/F_{n-1}\max \uparrow L_n\right],

where F_m := \bigcup_{k≤m} E(PC_k) and F_− := Id_X. Indeed, assuming \mathcal{P}_0,\ldots,\mathcal{P}_{n-1} are defined and coherent, F_{n-1} is a \rho-finite Borel equivalence relation, so applying Corollary 7.17 to \mathcal{S}[F_{n-1}]\left[\rho/F_{n-1}\max \uparrow L_n\right] with p := 1 yields a desired \mathcal{P}_n. (We will only use the saturation of \mathcal{P}_n.)

Claim 8.16. D_n := \text{dom}(\mathcal{P}_n) is a \mu-hyperfinitizing cut for G.

Proof of Claim. It is enough to show that D_n/F_{n-1} is a \mu/F_{n-1}-hyperfinitizing cut for G/F_{n-1} because the quotient map (X \setminus D_n) ↠ (X \setminus D_n)/F_{n-1} is a Borel reduction of E_G to E_{G/F_{n-1}}, where G_n := G_{−D_n}.

By Theorem 8.4, it is enough to show that (G_n)/F_{n-1} has finite \rho/F_{n-1}-visibility. But if it had infinite visibility, there would be an F_{n-1}-invariant U ∈ [X]^{0,∞}_G that admits F_{n-1}-invariant finite (G/F, \rho/F)-visible neighborhoods V ⊆ X \setminus D_n with arbitrarily large \rho(V)/\rho(U). The hypothesis of Lemma 8.14 implies that there are such V with the additional property that any other (G, \rho)-visible neighborhood of U containing V is in \mathcal{S}. Thus, taking V large enough so that \rho(V)/\rho(U) ≥ L_n implies that V ∈ \mathcal{S}[F_{n-1}]\left[\rho/F_{n-1}\max \uparrow L_n\right], contradicting the maximality of \mathcal{P}_n within \mathcal{S}[F_{n-1}]\left[\rho/F_{n-1}\max \uparrow L_n\right].

Let F_∞ := \bigcup_n F_n and D_∞ := \lim sup_n D_n := \{x ∈ X : x ∈ D_n for infinitely many n ∈ \mathbb{N}\}.

Claim 8.17. D_∞ is E_G-invariant.

Proof of Claim. Suppose not, so there are G-adjacent points x ∈ D_∞ and y ∈ X \setminus D_∞. Let N ∈ \mathbb{N} be large enough so that y \notin D_n for all n ≥ N; thus, V := [y]_{F_{∞}} = [y]_{F_N}, so V is \rho-finite. Because [x]_{F_{∞}} is \rho-infinite, taking n ≥ N large enough guarantees that

\rho([x]_{F_n}) ≥ L_n \cdot \rho(V).

Because, by the definition of \mathcal{P}_n, we also have

\rho([x]_{F_n}) ≥ L_n \cdot \max_{\rho/F_{n-1}} [x]_{F_n}

it follows that W := [x]_{F_n} \cup V is a (G/F_{n-1}, \rho/F_{n-1})-visible neighborhood of any U ∈ Max_{\rho/F_{n-1}} ([x]_{F_n} \cup V).
with \(\rho_{\mathcal{F}_{n-1}}^\text{max}(W) \geq L_n\); moreover, the same holds for any \((G_{\mathcal{F}_{n-1}}, \rho_{\mathcal{F}_{n-1}})\)-visible neighborhood of \(U\) containing \(W\), so \(W \in \mathcal{S}[\mathcal{F}_{n-1}]\left[\rho_{\mathcal{F}_{n-1}}^\text{max} \uparrow L_n\right]\), contradicting saturation.

\[\bullet\]

**Claim 8.18.** \(D_\infty\) is \(\mu\)-conull.

**Proof of Claim.** Suppose not, so \(Z := X \setminus D_\infty\) has positive measure. By Claim 8.17 and the hypothesis of Lemma 8.14, \(G|_Z\) is not \(\alpha\)-hyperfinite, so \(\alpha := \text{hvp}_\mu(G|_Z) > 0\) by Theorem 10.10. By Claim 8.16, \(\mu(D_n \cap Z) \geq \alpha\) for each \(n\), so, by the measure-compactness lemma, \(\mu(D_\infty \cap Z) \geq \alpha > 0\), contradicting our assumption that \(D_\infty \cap Z = \emptyset\).

Thus, we can take \(n \in \mathbb{N}\) large enough so that the set

\[X' := \{x \in X : \exists k \leq n \ [x]_{\mathcal{F}_n} \in \mathcal{P}_k\}\]

is \(\mu\)-co-\(\varepsilon\). Noting that \(X'\) is \(\mathcal{F}_n\)-invariant and for each \(x \in X'\), there is \(k \leq n\) such that

\[\rho_{\mathcal{F}_n}^\text{max}([x]_{\mathcal{F}_n}) \geq \rho_{\mathcal{F}_{k-1}}^\text{max}([x]_{\mathcal{F}_n}) \geq L_k \geq L,\]

we conclude the proof of Lemma 8.14 by letting \(\mathcal{P}\) be the collection of all \(\mathcal{F}_n\)-classes contained \(X'\).

\(\square\)(Lemma 8.14)

9. **Asymptotic averages along a graph**

If Theorem 2.1 is indeed true, then for a.e. \(x \in X\), there must be arbitrarily large finite \(G\)-connected sets containing \(x\) over which the average of \(f\) is arbitrarily close to 0. To verify this, we look at the set of all reals in general are achievable in this manner, thus defining a new invariant developed in this section.

Throughout this section, let \(G\) be a locally countable (abstract) graph on a set \(X\) and let \(\rho : E_G \rightarrow \mathbb{R}^+\) a cocycle. We also let \(f : X \rightarrow \mathbb{R}\) be a bounded function.

9.A. **For an abstract graph**

[MT17, Definition 8.2] introduces the set \(A^f[G]\) of asymptotic \(w\)-weighted means along \(G\) in the \(G\)-connected component \([x]_{E_G}\), where \(w : X \rightarrow \mathbb{R}^+\) is a weight-function. This set is independent of the representative \(x\) of the \(G\)-connected component [MT17, Proposition 8.3] and it is a closed interval when \(w\) is a bounded function [MT17, Proposition 8.5]. Here, we generalize this definition to arbitrary cocycles on \(E_G\).

**Definition 9.1.** For a \(G\)-connected set \(B\) and \(x \in B\), we call \(r \in \mathbb{R}\) a \((G, \rho)\)-asymptotic averages of \(f\) at \(x\) over \(B\) if there are \(\rho^x\)-arbitrarily large finite \(G\)-connected sets \(V \subseteq B\) containing \(x\) with \(A^f[V]\) arbitrarily close to \(r\); more precisely, for every \(\varepsilon > 0\) and \(L > 0\), there is a finite \(G\)-connected set \(V \subseteq B\) containing \(x\) with \(\rho^x(V) \geq L\) and \(A^f[V] \approx_\varepsilon r\). We denote by \(A^f[G|_B](x)\) the set of all such \(r \in \mathbb{R}\).

**Remark 9.2.** For a bounded block \(B = B(x, \alpha)\), we note that the restriction of \(\rho\) to \(B\) is a coboundary, being the differential of the weight function \(w_B : B \rightarrow \mathbb{R}^+\) defined by \(y \mapsto \rho^x(y)\); note that \(w_B\) is bounded above by \(\alpha\). It is easy to see that the definition of a \((G, \rho)\)-asymptotic average over \(B\) as above coincides with that of an asymptotic \(w_B\)-weighted mean along \(G|_B\) as defined in [MT17, Definition 8.2]. Thus, the following three lemmas are just restatements of [MT17, 8.4 and 8.5] in our terms.
Lemma 9.3. For any bounded block \( B \) and \( x, y \in B \), \( A_f^\rho[G|_B](x) = A_f^\rho[G|_B](y) \).

Proof. There is a \( G \)-path connecting \( x \) and \( y \), whose effect on the averages of \( f \) over arbitrarily large sets is arbitrarily small. \( \square \)

Lemma 9.4 (Intermediate value property). Let \( U, V \in [X]_G^{\rho<\infty} \) be such that \( U \subseteq V \) and let
\[
\Delta := \frac{\|f\|_{V \setminus U}}{\max \rho(V \setminus U)} \max \rho(U).
\]
For every real \( r \) between \( A_f^\rho[U] \) and \( A_f^\rho[V] \), there is \( W \in [X]_G^{\rho<\infty} \) with \( U \subseteq W \subseteq V \) and \( A_f^\rho[W] \approx \Delta \) \( r \).

Proof. Firstly, we replace \( V \) with \( V' \in [X]_G^{\rho<\infty} \) such that \( U \subseteq V' \subseteq V \), \( V' \setminus U \) is finite, and \( \rho(V') \geq \rho(V) - \varepsilon \), where \( \varepsilon > 0 \) is small enough to guarantee \( A_f^\rho[V'] > A_f^\rho[V] - \Delta \). In other words, we may assume without loss of generality that \( V \setminus U \) is finite. Now, we can add the vertices of \( V \setminus U \) to \( U \) one-by-one, obtaining a finite sequence of \( G \)-connected supersets of \( U \) increasing up to \( V \). It remains to observe that adding one vertex can change the average by at most \( \Delta \). \( \square \)

Lemma 9.5. For any bounded block \( B \) and \( x \in B \), \( A_f^\rho[G|_B](x) \) is closed and convex; thus, it is a closed subinterval of \([-\|f\|_\infty, +\|f\|_\infty]\).

Proof. The closedness follows from the compactness of \([-\|f\|_\infty, +\|f\|_\infty]\) and the very definition of \( A_f^\rho[G|_B] \). As for convexity, it follows from Lemma 9.4, because, in its notation, the numbers \( \max \rho(V \setminus U) \) are universally bounded\(^{17}\) for all \( \rho \)-finite \( V \subseteq B \), so by taking \( U \) arbitrarily large, we can make \( \Delta \) arbitrarily small. \( \square \)

Definition 9.6. We call \( r \in \mathbb{R} \) a \((G, \rho)\)-visible asymptotic average of \( f \) at \( x \) if it is a \((G, \rho)\)-asymptotic average of \( f \) over some bounded block \( B \ni x \).

Proposition 9.7 (Invariance). The map \( x \mapsto A_f^\rho[G](x) \) is \( E_G \)-invariant.

Proof. This is immediate from Lemma 9.3 the amalgamation property (8.2.c) of bounded blocks. \( \square \)

Proposition 9.8 (Convexity). For any \( x \in X \), \( A_f^\rho[G](x) \) is an increasing union of sets of the form \( A_f^\rho[G|_B](x) \), where \( B \) is a bounded block. In particular, it is a convex subset of \([-\|f\|_\infty, +\|f\|_\infty]\).

Proof. By definition, \( A_f^\rho[G](x) \) is a union of sets of the form \( A_f^\rho[G|_B](x) \), which is directed due to the (8.2.c) amalgamation property of bounded blocks. Thus, we can arrange it to be increasing due to the countability of \( B \). The convexity of \( A_f^\rho[G](x) \) is then due to Lemma 9.5. \( \square \)

\(^{17}\)This is the main reason why, in the definition of asymptotic averages, we allow taking limits only within a bounded block.
9.B. For measurable graphs

Equipping $X$ with a standard Borel structure, we now suppose further that $G, \rho,$ and $f$ are Borel, and we let $\mu$ be a $\rho$-invariant Borel probability measure on $X$.

Encoding intervals in $\mathbb{R}$ as points in $(\{\text{open, closed}\} \times \mathbb{R})^2$, we equip the set $\mathcal{I}$ of all intervals with a natural standard Borel structure.

**Proposition 9.9.** The map $x \mapsto \hat{A}_f^\rho[G](x) : X \to \mathcal{I}$ is Borel.

**Proof.** The fact that the image is in $\mathcal{I}$ is by Proposition 9.8. The Borelness follows by the definition of visible asymptotic averages via the Luzin–Novikov uniformization, of course. □

As mentioned in the beginning of this section, the main purpose of defining $\hat{A}_f^\rho[G]$ is to ensure that $\int_X f d\mu \in \hat{A}_f^\rho[G]$ a.e. This would only be useful if we could build a prepartition with arbitrarily large domain (of measure $1 - \varepsilon$) whose every cell $P$ is finite, $G$-connected, and $A_f^\rho[P]$ is in $\hat{A}_f^\rho[G]$ or at least arbitrarily close to it. We prove this next.

**Notation 9.10.** For $r \in \mathbb{R}$, $A \subseteq \mathbb{R}$, and $\varepsilon > 0$, we write $r \in \varepsilon A$ to mean that $\text{dist}(r, A) < \varepsilon$.

**Corollary 9.11.** Suppose that $G$ is $\mu$-nowhere hyperfinite. For every $\varepsilon > 0$, there is a Borel prepartition $\mathcal{P} \subseteq [X]^{\rho<\infty}_G$ with a $\mu$-co-$\varepsilon$ domain such that for each $P \in \mathcal{P}$, $A_f^\rho[P] \in \varepsilon \hat{A}_f^\rho[G](P)$.

**Proof.** Let $S$ denote the collection of all $U \in [X]^{\rho<\infty}_G$ satisfying $A_f^\rho[U] \in \varepsilon \hat{A}_f^\rho[G](U)$. It remains to show that $S$ satisfies the hypothesis of Lemma 8.14. But this is by the very definition of block-asymptotic averages. Indeed, for any $U \in [X]^{\rho<\infty}_G$, suppose towards a contradiction that there are visible neighborhoods $V$ of $U$ of large $\rho$-ratio that are not in $S$. Being visible neighborhoods of $U$, these $V$ are all contained in a single bounded block $B \supseteq U$, e.g. $B^\rho(x, \rho^x(U))$ for any $x \in U$. The compactness of $[-\|f\|_\infty, \|f\|_\infty]$ extracts a sequence $(V_n)$ with all $V_n$ contained in $B$ such that $\rho^{\max}(V_n) > n$ and $\lim_n A_f^\rho[V]$ exists and is outside of $\hat{A}_f^\rho[G](x)$, a contradiction. □

**Remark 9.12.** For a bounded weight function, instead of a cocycle, a stronger version of Corollary 9.11 holds: the requirement of $\mu$-nowhere hyperfiniteness is unnecessary and the prepartition in the conclusion has a conull domain. This is proven in [Miller-Tserunyan, 8.8] and it is due to the fact that in this case of an actual bounded weight function instead of a cocycle, the analogue of Lemma 8.14 is much easier to prove and has a stronger conclusion.

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10. Hyperfiniteness in terms of cuts

Let $X$ be a standard Borel space and let $G$ a locally countable Borel graph on it.

10.A. Vanishing sequences of (hyper)finitizing cuts

**Definition 10.1.** Call a subgraph $H \subseteq G$ a finitizing (resp. hyperfinitizing) edge-cut for $G$ if $G \setminus H$ is component-finite (resp. hyperfinite).

Call a sequence of sets vanishing if it is decreasing and has empty intersection.
Proposition 10.2. A locally countable Borel graph \( G \) is hyperfinite if and only if it admits a vanishing sequence \((H_n)\_n\) of finitizing Borel edge-cuts.

Proof. \( \Rightarrow \): For a witness \((F_n)\_n\) to the hyperfiniteness of \( E_G \), \( G_n := G \cap F_n \) is component-finite and \( G = \bigsqcup \_n G_n \), so the graphs \( G \setminus G_n \) are vanishing finitizing edge-cuts for \( G \).

\( \Leftarrow \): For each \( n \geq 1 \), \( G_n := G \setminus H_n \) is component-finite and \( G = \bigsqcup \_n=1 \infty G_n \).

Definition 10.3. Call a set \( V \subseteq X \) a finitizing (resp. hyperfinitizing) vertex-cut for \( G \) if \( G_{-V} := G abide \) is component-finite \( \) and \( G \setminus G_{-V} \) is a vanishing sequence of finitizing (resp. hyperfinitizing) vertex-cuts. The same is true for edge-cuts for \( G \).

An obvious connection between vertex-cuts \( V \) and edge-cuts \( H \) is via maps the \( H \mapsto \text{proj}_0(H) \) and \( V \mapsto [V, X]_G = \text{proj}_0^{-1}(V) \cap G \).

Observation 10.4. If \((V_n)\) is a vanishing sequence of (hyper)finitizing vertex-cuts for \( G \), then the graphs \( [V_n, X]_G \) form a vanishing sequence of (hyper)finitizing edge-cuts for \( G \).

In the reverse direction, we have:

Observation 10.5. Suppose that \( G \) is locally finite. Then, if \((H_n)\) is a vanishing sequence of (hyper)finitizing edge-cuts for \( G \), then the sets \( \text{proj}_0(H_n) \) form a vanishing sequence of (hyper)finitizing vertex-cuts for \( G \).

The local finiteness assumption here cannot be dropped: the sets \( \text{proj}_0(H_n) \) may not form a vanishing sequence as \([\text{MT}17, \text{Example 9.11}]\) shows.

10.B. Price of (hyper)finitizing

Let \( \mu \) and \( \nu \) be finite Borel measures on \( X \) and \( G \), respectively.

Definition 10.6. The finitizing edge-price (with respect to \( \nu \)) and the finitizing vertex-price (with respect to \( \mu \)) of \( G \) are the quantities:

\[
\begin{align*}
\text{fep}_\nu(G) & := \inf \{ \nu(H) : H \subseteq G \text{ is Borel finitizing edge-cut for } G \}, \\
\text{fvp}_\mu(G) & := \inf \{ \mu(V) : V \subseteq X \text{ is a Borel finitizing vertex-cut for } G \}.
\end{align*}
\]

Replacing “finitizing” with “hyperfinitizing” in the above definitions, we obtain hyperfinitizing vertex and edge prices denoted by \( \text{hvp}_\mu(G) \) and \( \text{hep}_\nu(G) \).

We characterize \( \mu \)-hyperfiniteness in terms of these notions. First, recall a basic lemma:

Lemma 10.7. For a measure space \((Y, M, \lambda)\) with \( \lambda \) finite, if a collection \( S \subseteq \mathcal{M} \) is closed under countable unions and contains sets of arbitrarily small \( \lambda \)-measure, then \( S \) admits a \( \lambda \)-vanishing sequence.

Proof. For each \( k \in \mathbb{N} \), take \( A_k \in S \) with \( \lambda(A_n) < 2^{-k} \). Then the sets \( B_n := \bigcup \_k>n A_k \) form a \( \lambda \)-vanishing sequence in \( S \).

This immediately implies:

Observation 10.8. If \( \text{fvp}_\mu(G) = 0 \) (resp. \( \text{hvp}_\mu(G) = 0 \)), then there is a \( \mu \)-vanishing sequence of finitizing (resp. hyperfinitizing) vertex-cuts. The same is true for edge-cuts for with respect to \( \nu \).

Proposition 10.9. For any locally countable Borel graph \( G \) on a standard Borel space \( X \) and any finite Borel measure \( \nu \) on \( G \), \( \text{hep}_\nu(G) = \text{fep}_\nu(G) \).

\( ^{18} \)We call a sequence \((A_n)\) of subsets of \( X \) \( \lambda \)-vanishing if it is decreasing and its intersection is \( \lambda \)-null.
Proof. We fix $\varepsilon > 0$ and show that $\text{hep}_\nu(G) < f\text{ep}_\nu(G) + \varepsilon$. Let $H_1$ be a hyperfinishing Borel edge-cut for $G$ with $\nu(H_1) < \text{hep}_\nu(G) + \frac{\varepsilon}{2}$. By Proposition 10.2 and the finiteness of $\nu$, $G \setminus H_1$ admits a finitizing Borel edge-cut $H_2$ with $\nu(H_2) < \frac{\varepsilon}{2}$, so $H_1 \cup H_2$ is a finitizing Borel edge-cut of $G$ with $\nu(H_1 \cup H_2) < f\text{ep}_\nu(G) + \varepsilon$. □

To have a connection between the $\mu$-hyperfiniteness of $G$ and $\nu$, we suppose that $\nu$ is a lift of $\mu$, i.e. $\text{proj}_0^* \nu \ll \mu$. For example, writing $G$ as a countable union of Borel maps $\gamma_n : X \to X$ (by the Luzin–Novikov theorem), we define: for a Borel set $A \subseteq G$,

$$\nu(A) := \sum_{n \geq 1} 2^{-n} \int_X 1_A(x, \gamma_n x) d\mu(x).$$

**Theorem 10.10.** For a locally countable Borel graph $G$ on a standard Borel space $X$, an $E_G$-quasi-invariant Borel probability measure $\mu$ on $X$, and a lift $\nu$ of $\mu$ to a finite Borel measure on $G$, the following are equivalent:

1. $G$ is $\mu$-hyperfinite.
2. $h\nu_\mu(G) = 0$.
3. $\text{hep}_{\nu}(G) = 0$.
4. $f\text{ep}_{\nu}(G) = 0$.

**Proof.** (10.10.1) $\Rightarrow$ (10.10.2) is trivial, (10.10.2) $\Rightarrow$ (10.10.3) is by $\text{proj}_0^* \nu \ll \mu$, (10.10.3) $\Rightarrow$ (10.10.4) is by Proposition 10.9, and (10.10.4) $\Rightarrow$ (10.10.1) is by Observation 10.8, Proposition 10.2, and $\mu \ll \text{proj}_0^* \nu$. □

11. **Proof of Theorem 2.1**

This section is entirely devoted to the proof of Theorem 2.1. We let $X, \mu, G, \rho, f$, and $\varepsilon$ be as in the statement of the theorem and, by subtracting the mean, we assume without loss of generality that $\int f d\mu = 0$, yet $\|f\|_1 > 0$.

**Claim 11.1.** We may assume without loss of generality that $f \in L^\infty(X, \mu)$.

**Proof of Claim.** Taking $\delta := \frac{\varepsilon}{2}$ and $g \in L^\infty(X, \mu)$ with $\|f - g\|_1 < \delta^2$, we apply Theorem 2.1 to $g$ and $\delta$ and let $F$ be as in the conclusion. Lemma 4.3 now gives

$$\|A^\rho_f[F]|_{X'} - A^\rho_k[F]|_{X'}\|_\infty \leq \frac{1}{\delta}\|f - g\|_1 = \delta$$

for an $F$-invariant $\mu$-co-$\delta$ set $X'$, so $A^\rho_f[F](x) \approx \mu$ $f d\mu$ for a $\mu$-co-$\varepsilon$ set of $x \in X$. $\Box$

For $\delta \geq 0$, denote $I_\delta := (-\delta, \delta)$ and for any open interval $I := (a, b)$, put

$$I^- := (-\infty, a], \quad \text{and} \quad I^+ := [b, +\infty).$$

By the ergodicity of $G$, the map $x \mapsto \bar{A}^\rho_f|G|(x)$ is constant a.e. because it is $E_G$-invariant; thus, we drop $x$ from the notation.

Due to the equivalence of Theorems 2.1 and 1.4, we may assume without loss of generality that $G$ is not $\mu$-hyperfinite and hence $\mu$-nowhere hyperfinite by ergodicity.
11.A. Cutting one side of the set of asymptotic averages

Note that taking a quotient of $G$ by a $G$-connected finite Borel subequivalence relation $F \subseteq E_G$ results in a graph $G/F$ with a smaller set of visible asymptotic averages, i.e.

$$\tilde{A}^\rho _f [G : F] := \tilde{A}^\rho _f [G/F] \subseteq \tilde{A}^\rho _f [G].$$

Having the ability (Corollary 9.11) to build prepartitions whose each cell $P$ has $A^\rho _f [P]$ arbitrarily close to $\tilde{A}^\rho _f [G]$, it is natural to aim at building a special $F$ such that quotienting out by it shrinks the set of asymptotic averages around 0 arbitrarily tightly. In fact, a weaker conclusion is enough due to the following easy fact.

**Lemma 11.2** (Cutting one side is enough). For any $\varepsilon > 0$, there is an (explicit) $\delta > 0$ such that for any $G$-connected finite Borel equivalence relation $F$ and any sign $s \in \{+, -\}$, if $A^\rho _f [F] \in I_\delta \cup I^s_\delta$ with $1 - \delta$ probability, then $A^\rho _f [F] \in I_\varepsilon$ with $1 - \varepsilon$ probability.

**Proof.** Given $\varepsilon > 0$, we take $\delta := \frac{\varepsilon^2}{\|f\|\infty + 1}$ and fix arbitrary $F$ and $s$ as in the hypothesis. We assume without loss of generality that $s = +$ and we let

$$A := \{x \in X : A^\rho _f [F] (x) \not\in I_\delta \cup I^s_\delta \},$$

$$B := \{x \in X \setminus A : A^\rho _f [F] (x) \not\in I_\varepsilon \} = \{x \in X : A^\rho _f [F] (x) \in I^s_\varepsilon \},$$

where the last equality is because $I_\delta \subseteq I_\varepsilon$. Observing that $A$ and $B$ are $F$-invariant and Borel, we need to show that $\mu(B) < \varepsilon$. To this end, using (4.2.a), we compute:

$$0 = \int_X f \, d\mu = \int_X A^\rho _f [F] \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu + \int_{X \setminus (A \cup B)} f \, d\mu \geq -\|f\|\infty \cdot \mu(A) + \varepsilon \cdot \mu(B) - \delta \cdot \mu(X \setminus (A \cup B)) \geq -\|f\|\infty \cdot \delta + \varepsilon \cdot \mu(B) - \delta \cdot \left(1 - \delta - \mu(B) \right) = \mu(B) \cdot (\varepsilon + \delta) - \delta \cdot (\|f\|\infty + 1 - \delta),$$

so

$$\mu(B) = \delta \cdot \frac{\|f\|\infty + 1 - \delta}{\varepsilon + \delta} < \delta \cdot \frac{\|f\|\infty + 1}{\varepsilon} = \varepsilon.$$  

Thus, if for every $\delta > 0$, there was a $G$-connected finite Borel equivalence relation $F$ and a sign $s \in \{+, -\}$ with $\tilde{A}^\rho _f [G : F] \subseteq I_\delta \cup I^s_\delta$, then Corollary 9.11 applied to the quotient by $F$ would yield a $G$-connected finite Borel equivalence relation $F' \supseteq F$ with $A^\rho _f [F'] \in I_{2\delta} \cup I^s_{2\delta}$ on a $\mu$-co-$\delta$ set. Choosing $\delta$ sufficiently small, Lemma 11.2 would imply that $A^\rho _f [F'] \in I_\varepsilon$ holds with probability $1 - \varepsilon$, so $H := F' \cap G$ satisfies the conclusion of Theorem 2.1. Thus, without loss of generality, we may assume the following hypothesis, where for an interval $I \subseteq \mathbb{R}$, we say that a set $S \subseteq \mathbb{R}$ spills over both sides of $I$ if both $S \cap I^+$ and $S \cap I^-$ are nonempty.

**Hypothesis 11.3.** There is a $\delta > 0$ such that for any $G$-connected finite Borel equivalence relation $F$, $\tilde{A}^\rho _f [G : F]$ spills over both sides of $I_\delta$.

We exploit this hypothesis and the $\mu$-nowhere hyperfiniteness of $G$ via packed prepartitions.
11.B. Domains of packed prepartitions are large

For \( \lambda > 0 \), we call \( U \in [X]_G^{<\infty} \) \( \lambda \)-central (resp., \( \lambda \)-negative, \( \lambda \)-positive) if \( A_f^\rho[U] \in I_\lambda \) (resp., \( I_\lambda^-, I_\lambda^+ \)).

For a \( G \)-connected \( \rho \)-finite equivalence relation \( F \) and \( L > 0 \), let \( S(f, \lambda, L, F) \subseteq [X]_G^{<\infty} \) denote the collection of all \( \lambda \)-central \( F \)-invariant \( U \in [X]_G^{<\infty} \) of \( \rho/F \)-ratio at least \( L \), where by \( \rho/F \)-ratio of \( U \) we mean that of \( U/F \). We omit writing \( F \) if it is just the identity relation.

**Lemma 11.4** (Finitizing visibility). Let \( \lambda, L > 0 \) and \( p := \frac{\lambda}{\|f\|_\infty} \). For any prepartition \( \mathcal{P} \) \( p \)-packed within \( S(f, \lambda, L) \), if \( \tilde{A}_f^\rho[G : E(\mathcal{P})] \) spills over both sides of \( I_\lambda \) on every \( G \)-connected component, then \( G_{\text{dom}(\mathcal{P})} \) has finite \( \rho \)-visibility.

**Proof.** Towards the contrapositive, we suppose that \( G_{\text{dom}(\mathcal{P})} \) does not have finite \( \rho \)-visibility and aim at showing that \( \mathcal{P} \) is not \( p \)-packed within \( S(f, \lambda, L) \). Let \( x \in X \setminus \text{dom}(\mathcal{P}) \) be such that \( B := B_{G_{\text{dom}(\mathcal{P})}}^\rho(x) \) is \( \rho \)-infinite.

If there is \( V \in S(f, \lambda, L) \) that is entirely contained in \( B \), then \( \mathcal{P} \) is not even maximal within \( S(f, \lambda, L) \) and we are done, so suppose there isn’t such a \( V \). This, together with the intermediate value property (Lemma 9.4), implies that \( x \) cannot have both \( \lambda \)-positive and \( \lambda \)-negative visible neighborhoods within \( B \) of arbitrarily large \( \rho \)-ratio. Thus, all large enough visible neighborhoods of \( x \) within \( B \) must have the same \( \lambda \)-sign. For concreteness, suppose that they are \( \lambda \)-positive.

Because \( \tilde{A}_f^\rho[G : E(\mathcal{P})](x) \) intersects \( I_\lambda^+ \), there is a \( \lambda \)-negative \( E(\mathcal{P}) \)-invariant \( U \in [X]_G^{<\infty} \) containing \( x \) with \( \rho^\max(U) \geq L \) and \( \frac{\|f\|_\infty \rho(x)}{\rho(U)} < \lambda \). The latter ensures, by the intermediate value property again, that there is \( W \in [B]_G^{<\infty} \) disjoint from \( U \) but \( G \)-adjacent to \( U \) such that

\[
0 \leq A_f^\rho[U \cup W] < \lambda.
\]

(11.5)

Furthermore, because \( W \subseteq B, \max_p W \leq \rho(x) \leq \max_p U \), so \( \rho^\max(U \cup W) \geq \rho^\max(U) \geq L \) and hence, \( U \cup W \in S(f, \lambda, L) \). Thus, to show that \( \mathcal{P} \) is not \( p \)-packed, it is enough to show that \( \rho(W) \geq \rho(U) \). To this end, by (4.1.a),

\[
\rho(W) \cdot A_f^\rho[W] = \left( \rho(U) + \rho(W) \right) \cdot A_f^\rho[U \cup W] - \rho(U) \cdot A_f^\rho[U] \\
\geq \left( \rho(U) + \rho(W) \right) \cdot 0 - \rho(U) \cdot (-\lambda) = \lambda \cdot \rho(U).
\]

As mentioned above, \( W \) is not \( \lambda \)-central, and it cannot be \( \lambda \)-negative either because then \( U \cup W \) would be \( \lambda \)-negative, contradicting (11.5). Therefore, \( W \) is \( \lambda \)-positive, in particular, \( A_f^\rho[W] > 0 \). It now follows from the last calculation that

\[
\rho(W) > \frac{\lambda}{A_f^\rho[W]} \cdot \rho(U) \geq \frac{\lambda}{\|f\|_\infty} \cdot \rho(U) \geq p \cdot \rho(U).
\]

\[ \square \]

In conjunction with Theorem 8.4, Lemma 11.4 gives:

**Corollary 11.6.** Let \( \lambda, L > 0 \) and \( p := \frac{\lambda}{\|f\|_\infty} \) positive. For any Borel prepartition \( \mathcal{P} \) \( p \)-packed within \( S(f, \lambda, L) \), if \( \tilde{A}_f^\rho[G : E(\mathcal{P})] \) spills over both sides of \( I_\lambda \) on every \( G \)-connected component, then \( \text{dom}(\mathcal{P}) \) is a hyperfinitizing Borel vertex-cut; in particular, \( \mu(\text{dom}(\mathcal{P})) \geq \text{fvp}_\mu(G) \).
11.C. Iteration via measure-compactness

To construct a desired component-finite subgraph $H$, we first obtain a coherent sequence of saturated prepartitions $(\mathcal{P}_n)$ that contain larger and larger and more and more central sets while becoming more and more packed. We will show that putting together enough finite-many of these $\mathcal{P}_n$ yields a desired $H$.

Fix $\delta > 0$ as in Hypothesis 11.3 and for each $n \geq 1$, put

$$\lambda_n := 3^{-n}\delta$$
$$L_n := 4^n$$
$$p_n := \frac{\lambda_{n+2}}{\|f\|_\infty + \lambda_{n+1}}.$$  

**Remark 11.7.** All we need below is that $\lim_{n} \lambda_n L_n = \infty$ (this guarantees (11.11), and hence, Claim 11.13) and that $(\lambda_n)$ decays exponentially to 0 (used in Claim 11.14). The choice of $(p_n)$ is made to yield a contradiction in the proof of Claim 11.15.

Ignoring $\rho$-deficient (hence $\mu$-null) sets, we recursively apply the (relative) packing and saturation lemma (Corollary 7.17) to get a coherent sequence $(\mathcal{P}_n)_{n=1}^\infty$ of prepartitions such that, for each $n \geq 1$, $\mathcal{P}_n$ is saturated and $p_n$-packed within $\mathcal{S}(f, \lambda_n, L_n, F_{n-1})$, where $F_0 := \text{Id}_X$ and

$$F_{n-1} := \bigcup_{k \leq n-1} E(\mathcal{P}_k).$$

Because $\lambda_n \leq \delta$, Hypothesis 11.3 implies that $\mathcal{A}_f^\rho[\mathcal{G} : E(\mathcal{P}_n)]$ spills over both sides of $I_{\lambda_n}$, so Corollary 11.6 implies that $D_n := \text{dom}(\mathcal{P}_n) \supseteq \text{fvp}_\rho(G) > 0$, so by measure-compactness (Observation 11.8 below), $D_\infty := \limsup_n D_n$ has positive measure.

**Observation 11.8 (Measure-compactness).** In a finite measure space $(X, \mu)$, for measurable sets $\{D_n\}_{n \in \mathbb{N}}$, $\mu(\limsup_n D_n) \geq \limsup_n \mu(D_n)$, where

$$\limsup_n D_n := \{x \in X : \exists^\infty n \ x \in D_n\} = \bigcap_{N \geq N} \bigcup_{n \geq N} D_n.$$  

In the next subsection, we show that $D_\infty$ is actually conull. Granted this, the proof of Theorem 2.1 is completed as follows: for large enough $m < n$, $\lambda_m < \epsilon$ and every point $x$ in a co-$\frac{\epsilon}{2}$ set belongs to $D_k$ for some $k \in [m, n]$, so $[x]_{\mathcal{F}_n}$ is $I_{m-1}$-central, hence $F_n$ almost satisfies the conclusion of Theorem 2.1, except that its classes are only $\rho$-finite, but they may be infinite. But then we can choose, uniformly in a Borel fashion, a large enough finite $G$-connected subset from each $F_n$-class, thus defining a finite Borel $G$-connected subequivalence relation $F \subseteq F_n$ such that every point $x$ in a co-$\epsilon$ set, $[x]_F$ is $I_{m-1}$-central. Now this $F$ fully satisfies the conclusion of Theorem 2.1.

11.D. The conullness of $D_\infty$ reduces to building a $\rho$-deficient flow

Suppose towards a contradiction that $X \setminus D_\infty$ has positive measure, so the inner-boundary $\partial^\text{in}_G(D_\infty)$ is also of positive measure.

**Notation 11.9.** For $x \in X$ and $n \in \mathbb{N}$, let $k_n(x)$ denote the largest number $m \leq n$ such that $x \in D_m$. Because the map $x \mapsto k_n(x)$ is $F_n$-invariant, we also put $k_n(U) := k_n(x)$ for any $F_n$-class $U$ and $x \in U$. 

39
Proof of Claim. Due to (4.1.b) and (11.11), so
\[
\lambda_n \geq \rho([x]_{F_n}) \leq \rho([x]_{F_n}) \geq \rho([x]_{F_n})
\]
Therefore, for a large enough \( N \geq 1 \), the set \( \text{proj}_0(R_N) \subseteq \partial_{G}^\infty(D_\infty) \) is of positive measure and, for all \( n \geq N \),
\[
L_n \geq 8 \text{ and } \frac{\|f\|_\infty}{L_n} < \frac{1}{3} \lambda_n.
\]
Furthermore, we can choose such an \( N \) so that \( S := \text{proj}_0(R_N) \cap D_N \) has positive measure; this ensures that for any \( x \in S \) and \( n \geq N \), \( k_n(x) \geq N \). Finally, we put
\[
R := R_N \cap (S, X)_G.
\]
We will define a Borel \( \rho \)-flow \( \varphi \) on \([S]_{F_\infty}\) with Sources(\( \varphi \)) = \( S \) and no sinks, thus proving the \( \rho \)-deficiency of \([S]_{E_G}\), contradicting \( \mu(S) > 0 \), by Corollary 6.6.

11.E. Envisioning a canal for the flow

Our goal in this subsection is to prove Claim 11.15 for every \( F_\infty \)-class \( C \subseteq [S]_{F_\infty} \), so we fix such a \( C \) and let \( R(C) := \{ y \in X : \exists x \in C \text{ with } (x, y) \in R \} \).

Claim 11.13. There is a sign \( s_C \in \{+,-\} \) such that for every \( y \in R(C) \), \( A_f^\rho([y]_{F_\infty}) \in I_0^{s_C} \).

Proof of Claim. Suppose towards a contradiction that there are pairs \((x_-, y_-), (x_+, y_+) \in R\) with \( x_-, x_+ \in C \) (possibly equal) such that \( A_f^\rho([V_-]) \leq 0 \leq A_f^\rho([V_+]) \), where \( V_- := [y_-]_{F_\infty} \) and \( V_+ := [y_+]_{F_\infty} \).

Let \( n > N \) be large enough so that \( P := [x_-]_{F_n} = [x_+]_{F_n} \) and such that \( x_+ \in D_n \), ensuring that \( P \in \mathcal{P}_n \).

Without loss of generality, we suppose that \( A_f^\rho(P) < 0 \) (the argument is symmetric) and take \( P' := P \cup V_+ \). Because \((x_+, y_+) \in R \subseteq R_N \), the definition (11.10) and (11.12) of \( R_N \) implies that
\[
\max_{\rho/F_{n-1}} P \geq \max_{\rho/F_N} P \geq \rho([x_+]_{F_n}) \geq \rho(V_+),
\]
so \( P' \) is an \( F_{n-1} \)-invariant \( G \)-connected set with \( \rho/F_{n-1} \)-ratio at least that of \( P \), and hence at least \( L_n \). Finally, adding \( V_+ \) to \( P \) increases the average and by at most
\[
\frac{\|f\|_\infty \cdot \rho(V_+)}{\rho(P)} \leq \frac{\|f\|_\infty \cdot \rho(V_+)}{L_n \cdot \max_{\rho/F_{n-1}} P} \leq \frac{\|f\|_\infty}{L_n} < \lambda_n
\]
due to (4.1.b) and (11.11), so \( P' \) is still \( I_{\lambda_n} \)-central and hence is in \( S(f, \lambda_n, L_n, F_{n-1}) \), contradicting the saturation of \( \mathcal{P}_n \) within \( S(f, \lambda_n, L_n, F_{n-1}) \).

Suppose, for the sake of concreteness, that \( s_C = - \).

Claim 11.14. For each \( x \in S \cap C \) and each \( n > N \), \( A_f^\rho([x]_{F_n}) < -\frac{2}{3} \lambda_n = -2 \lambda_{n+1} \).

Proof of Claim. Because the \( \lambda_n \) are decreasing, it suffices to prove this claim when \( n = k_n(x) \) (recall that \( k_n(x) \geq N \)). Thus, \( P := [x]_{F_n} \in \mathcal{P}_n \).
Let \( V_\cdot \) be an \( F_\infty \)-class of a point \( y \) such that \((x,y) \in R\). As in the proof of Claim 11.13, \( \max_{\rho/F_{n-1}} \rho > \rho(V_\cdot) \), so \( P' := P \cup V_\cdot \) is \( F_{n-1} \)-invariant and its \( \rho/F_{n-1} \)-ratio is at least \( L_n \). Furthermore, by (4.1.b) and (11.11), adding \( V_\cdot \) to \( P \) perturbs the average by at most

\[
\frac{\|f\|_\infty \cdot \rho(V_\cdot)}{\rho(P)} \leq \frac{\|f\|_\infty \cdot \rho(V_\cdot)}{L_n \cdot \max_{\rho/F_{n-1}} \rho} \leq \frac{1}{3} \lambda_n.
\]

Moreover, our assumption of \( s_C = -\) implies that \( A^\rho_f(V_\cdot) < 0 \), so, if \( A^\rho_f(P) \geq -\frac{2}{3} \lambda_n \), then

\[-\lambda_n < A^\rho_f(P \cup V_\cdot) \leq A^\rho_f(P) < \lambda_n,
\]

and hence \( P' \) is also \( I_{\lambda_n} \)-central, contradicting the saturation of \( P_n \) within \( S(f, \lambda_n, L_n, F_{n-1}) \).

Claim 11.15. For each \( n \geq N \) and \( P \in \mathcal{P}_{n+1} \),

\[\rho(P \setminus ([S]_{F_n} \cap P)) \geq \frac{2}{13} \cdot \rho([S]_{F_n} \cap P).\]

Proof of Claim. Supposing otherwise, we will contradict the \( p_n \)-packedness of \( P_n \).

Putting \( U := [S]_{F_n} \cap P \), \( V := (D_n \cap P) \setminus U \), and \( W := P \setminus (U \cup V) \), we compute:

\[-\lambda_{n+1} \leq A^\rho_f[P] = \frac{1}{\rho(P)} \cdot \left( A^\rho_f[U] \cdot \rho(U) + A^\rho_f[V] \cdot \rho(V) + A^\rho_f[W] \cdot \rho(W) \right),
\]

so

\[-\lambda_{n+1} \cdot (\rho(U) + \rho(V) + \rho(W)) \leq A^\rho_f[U] \cdot \rho(U) + A^\rho_f[V] \cdot \rho(V) + A^\rho_f[W] \cdot \rho(W)
\]

\[
\left[ \text{by Claim 11.14} \right] \leq -2\lambda_{n+1} \cdot \rho(U) + \lambda_n \cdot \rho(V) + \|f\|_\infty \cdot \rho(W)
\]

and thus,

\[
\lambda_{n+1} \cdot \rho(U) \leq (\lambda_n + \lambda_{n+1}) \cdot \rho(V) + (\|f\|_\infty + \lambda_{n+1}) \cdot \rho(W)
\]

\[
\left[ \text{by the } p_n \text{-packedness of } P_n \right] \leq (\lambda_n + \lambda_{n+1}) \rho(V) + (\|f\|_\infty + \lambda_{n+1}) \cdot p_n \cdot (\rho(U) + \rho(V))
\]

\[
= (\lambda_n + \lambda_{n+1}) \cdot \rho(V) + \lambda_{n+2} \cdot (\rho(U) + \rho(V)).
\]

This gives

\[
(\lambda_{n+1} - \lambda_{n+2}) \cdot \rho(U) \leq (\lambda_n + \lambda_{n+1} + \lambda_{n+2}) \cdot \rho(V),
\]

and because \( \lambda_n = 9\lambda_{n+2} \) and \( \lambda_{n+1} = 3\lambda_{n+2} \), we finally get:

\[
\rho(V) \geq \frac{\lambda_{n+1} - \lambda_{n+2}}{\lambda_n + \lambda_{n+1} + \lambda_{n+2}} \cdot \rho(U) = \frac{(3 - 1)\lambda_{n+2}}{(9 + 3 + 1)\lambda_{n+2}} \cdot \rho(U) = \frac{2}{13} \cdot \rho(U).
\]

11.F. The set \( S \) flows away

Claim 11.15 is proven for an arbitrary \( F_\infty \)-class \( C \subseteq [S]_{F_\infty} \), so it is true for every \( P \in \mathcal{P}_{n-1} \) contained in \([S]_{F_\infty} \). This allows us to recursively define a sequence \((\varphi_n)_{n \geq N}\) of Borel \( \rho \)-flows with

\[\text{dom}(\varphi_n) \subseteq ([S]_{F_n} \cap D_{n+1}) \times (D_{n+1} \setminus [S]_{F_n}) \cap F_{n+1} \]

(11.16)

(hence pairwise disjoint), whose sum \( \varphi_\infty := \sum_{n \geq N} \varphi_n \) has no sinks, yet \( \text{Sources}(\varphi_\infty) = S \). This witnesses the \( \rho \)-deficiency of \( S \), thus finishing the proof. It remains to construct
(φ_n)_{n \geq N} and the construction is very similar to that of the proof of Lemma 7.7, with \frac{2}{13} instead of p.

Putting \( \bar{\phi}_n := \sum_{k=1}^n \phi_k \), we will maintain the following additional property:
\[
\text{for each } P \in \mathcal{P}_n, \quad \int_{\text{Sinks}(\bar{\phi}_n) \cap P} \inf \bar{\phi}_n \ d\rho \leq \frac{2}{13} \cdot \rho(P). \tag{11.17}
\]

Let \( \phi_N \) be the zero-\( \rho \)-flow and, for fixed \( n \geq N \), suppose \( \phi_N, \phi_{N+1}, \ldots, \phi_n \) are defined and satisfy (11.16) and (11.17).

We will define the \( \rho \)-flow \( \phi_{n+1} \) only on \( \text{dom}(\phi_n) \subseteq ([S]_{F_n} \cap D_{n+1}) \times (D_{n+1} \setminus [S]_{F_n}) \cap F_{n+1} \) using Lemma 6.9. First, we put
\[
U_n := [S]_{F_n} \cap D_{n+1},
\]
\[
V_n := [U_n]_{F_{n+1}} \setminus U_n,
\]
and define \( \omega_n : U_n \to [0,1] \) as follows: for each \( P \in \mathcal{P}_{n+1} \), \( \omega_n \mid_{U_n \cap P} := \frac{2}{13} \) if \( \bar{\phi}_n \) has no sinks in \( U_n \cap P \); otherwise, for \( x \in U_n \ cap P \),
\[
\omega_n(x) := 1_{\text{Sinks}(\bar{\phi}_n)}(x) \cdot \inf \bar{\phi}_n(x).
\]
Note that in the latter case, the inductive assumption (11.17) gives
\[
\int_{U_n \cap P} \omega_n \ d\rho = \int_{\text{Sinks}(\bar{\phi}_n) \cap U_n \cap P} \inf \bar{\phi}_n \leq \frac{2}{13} \cdot \rho(U_n \cap P).
\]
Thus, in either case,
\[
\int_{U_n \cap P} \omega_n \ d\rho \leq \frac{2}{13} \cdot \rho(U_n \cap P). \tag{11.18}
\]
By Claim 11.15,
\[
\rho(P \cap V_n) = \rho(P \setminus [S]_{F_n}) \geq 2 \cdot \rho([S]_{F_n} \cap P) = \frac{2}{13} \cdot \rho(U_n \cap P) \geq \int_{U_n \cap P} \omega_n \ d\rho,
\]
so Lemma 6.9 applies to \( F_{n+1} \mid_{\text{dom} \mathcal{P}_{n+1}}, U_n, V_n, \omega_n, \) and gives the desired \( \rho \)-flow \( \phi_{n+1} \). It remains to note that \( \text{Sinks}(\bar{\phi}_{n+1}) \cap [U_n]_{F_{n+1}} \subseteq V_n \), so, due to Lemma 6.9 and (11.18), for each \( P \in \mathcal{P}_{n+1} \),
\[
\int_{\text{Sinks}(\bar{\phi}_{n+1}) \cap P} \inf \bar{\phi}_{n+1} \ d\rho = \int_{V_n \cap P} \inf \bar{\phi}_{n+1} \ d\rho = \int_{U_n \cap P} \omega_n \ d\rho \leq \frac{2}{13} \cdot \rho(U_n \cap P) \leq \frac{2}{13} \cdot \rho(P).
\]
verifying (11.18) for \( \phi_{n+1} \). This completes the construction of \( \phi_n \) for \( n \geq N \). \( \square \) (Theorem 2.1)

**References**


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