A DESCRIPTIVE SET THEORIST'S PROOF OF THE POINTWISE ERGODIC THEOREM

ANUSH TSERUNYAN

Abstract. We give a short combinatorial proof of the classical pointwise ergodic theorem for probability measure preserving $\mathbb{Z}$-actions [Bir31]. Our approach reduces the theorem to a tiling problem: tightly tile each orbit by intervals with desired averages. This tiling problem is easy to solve for $\mathbb{Z}$ with intervals as tiles. However, it would be interesting to find other classes of groups and sequences of tiles for which this can be done, since then our approach would yield a pointwise ergodic theorem for such classes.

Let X be any set and $f : X \to \mathbb{R}$. For any finite nonempty subset $U \subseteq X$, we put

$$A_f U := \frac{1}{|U|} \sum_{y \in U} f(y),$$

and for a finite equivalence relation[1] F on X, define $A_f[F] : X \to \mathbb{R}$ by $A_f[F](x) := A_f[x]_F$.

Lemma 1 (Finite averages)**.** *For any measure preserving finite Borel equivalence relation F on a standard measure space (X, μ) and any $f \in L^1(X, \mu)$,*

$$\int f d\mu = \int A_f[F]\, d\mu.$$

Proof. For each $n \in \mathbb{N}$, restricting to the part of X where each F-class has size n, we may assume X is that part to begin with. Because each F-class is finite, (using the Luzin–Novikov countable section uniformization theorem [Kec95, 18.10]) there is a Borel automorphism T that induces F. Using the T-invariance of μ, we deduce

$$\int_X f(x) d\mu(x) = \frac{1}{n} \sum_{i<n} \int_X f(T^i x) d\mu(x) = \int_X A_f[F](x)\, d\mu(x). \qquad \square$$

Suppose from now on that (X, μ) is a standard probability space and let T be an aperiodic automorphism of (X, μ). Let $\leqslant_T$ be the partial order on X induced by T, that is: $x \leqslant_T y :\Leftrightarrow \exists n \in \mathbb{N}\, T^n x = y$. For $x, y \in X$, put $(x, y)_T := \{z \in X : x <_T z <_T y\}$ and call the sets of this form *T-intervals*; also, define $[x, y)_T$ and $(x, y]_T$ expectedly. We say that subset S of a T-orbit is *bi-infinite* if it has no minimum or maximum with respect to $\leqslant_T$.

Theorem 2 (Pointwise ergodic for ergodic actions)**.** *For ergodic T, $\lim_{n\to\infty} A_f[x, T^n x)_T = \int f\, d\mu$ a.e.*

Proof. Replacing f with $f - \int f d\mu$, we may assume that $\int f d\mu = 0$. We show that for a.e. $x \in X$, $\limsup_{n\to\infty} A_f[x, T^n x)_T \leqslant 0$ and an analogous argument shows that $\liminf_{n\to\infty} A_f[x, T^n x)_T \geqslant 0$.

Because the map $x \mapsto \limsup_{n\to\infty} A_f[x, T^n x)_T$ is T-invariant, ergodicity implies that it is constant a.e. Suppose towards a contradiction that that constant c is positive and put $\Delta := \frac{c}{2}$. Take $\varepsilon := \frac{\min(\Delta,1)}{8}$ and let $0 < \delta_f < \varepsilon$ be small enough so that any set $A \subseteq X$ of measure less than δ_f (call such sets *small*) supports less than ε mass of f, i.e., $\|f \cdot \mathbb{1}_A\|_1 < \varepsilon$. Let L be large enough so that the set

$$Z := \{x \in X : A_f[x, T^n x)_T < \Delta \text{ for all integers } n \in [1, L]\}$$

is small. Define a function $\ell : X \to \mathbb{N}$ by mapping x to the smallest $n \leqslant L$ such that $A_f[x, T^n x)_T \geqslant \Delta$, if $x \notin Z$, and to 1, otherwise. For each $x \in X$, put $I_x := [x, T^{\ell(x)} x)_T$ and call such T-intervals *tiles*. Say that a T-interval $I := [y, z)_T$ is *tiled* if it admits a partition (*tiling*) into tiles. It follows by induction on the length of I that such a partition, if exists, is unique because I_y has to be the tile containing y.

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[1]By a *finite equivalence relation* F, we mean that each F-class is finite.

Let $S \subseteq X$ be a nonnull set such that $\tilde{S} \coloneqq \bigcup_{i=1}^{L} T^{-i} S$ is small. By ergodicity, S is nonempty in a.e. orbit. Furthermore, the orbits in which S is nonempty but not bi-infinite allow for a choice of single point in a Borel fashion, so they form a null set and we ignore them. For each $x \in X$, denote by $s(x)$ the closest element of S to the left of x, i.e. $s(x) \in S$, $s(x) \leqslant_T x$, and $\big(s(x), x\big]_T \cap S = \emptyset$. Let $r(x)$ denote the $\leqslant_T$-least point $\geqslant_T x$ such that $[s(x), r(x))_T$ is tiled.

Define a finite equivalence relation F on X by setting $x \, F \, y$ if and only if $s(x) = s(y)$ and x, y are in the same tile of the unique tiling of $[s(x), r(x))_T$. For any $x \in X$, $[x]_F$ is a tile if $x \notin \tilde{S}$, and $[x]_F = \{x\}$ if $x \in Z$. Thus, F-classes of points in $X \setminus (\tilde{S} \cup Z)$ form an F-invariant set Y of measure $> 1 - 2\delta_f$ and for each $y \in Y$, $[y]_F = I_z$ for some $z \in X \setminus Z$, so $A_f[F](y) \geqslant \Delta$. Thus, Lemma 1 implies a contradiction:

$$0 = \int_X f d\mu \geqslant \int_Y f d\mu - 2\varepsilon = \int_Y A_f[F] d\mu - 2\varepsilon \geqslant \Delta(1 - 2\delta_f) - 2\varepsilon \geqslant \frac{\Delta}{2} > 0. \qquad \square$$

Remark 3. Another short proof of the pointwise ergodic theorem for $\mathbb{Z}$ is given by Keane and Petersen in [KP06]. The proof is analytic and has the advantage of not using any black box, whereas we do use the Luzin–Novikov uniformization theorem to keep the sets measurable[2]. The tiling is implicitly present in Keane–Petersen proof, but without turning it into an equivalence relation, so it is not clear how to adapt their proof to other shapes of tiles in other groups.

Our approach explicitly reduces the pointwise ergodic theorem to a tiling problem, which makes it interesting to consider this problem for other groups and sequences of tiles. If solved, our approach then would yield a pointwise ergodic theorem for those groups.

References

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(Anush Tserunyan) Department of Mathematics, University of Illinois at Urbana-Champaign, IL, 61801, USA

Email address: anush@illinois.edu

[2] Although, instead, we could easily observe that these sets are in the σ-ideal generated by analytic sets and use that these sets are universally measurable.