Poisson-Voronoi tessellations and fixed price for higher rank lattices

McGill DDC Seminar

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Background
## History of the IPVT

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<td>Budzinski, Curien, Petri</td>
<td>Description of the pointless Voronoi tessellation on $\mathbb{H}^2$</td>
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<tr>
<td>2023</td>
<td>D’Achille, Curien, Enriquez, Lyons, Unel</td>
<td>Construction of the ideal Poisson-Voronoi tessellation (IPVT) on $\mathbb{H}^d$</td>
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<td>(soon)</td>
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<td>Construction of the IPVT on a higher rank real semisimple Lie group $G$</td>
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$\frac{1}{\eta} = \text{Poisson point process on } X \text{ with intensity } \eta$

**Idea for $\mathbb{H}^2$**

IPVT construction
Horocones

We call the object on which the IPVT lives a horocone.

The horocone for $G = SL(2, \mathbb{R})$ and $X = \mathbb{H}^2$ is $G$ modded out by the subgroup of upper triangular matrices with ones on the diagonal, equivalently $\partial X \times \mathbb{R}$, equipped with Lebesgue measure.

Theorem (FMW)

Any nonamenable locally compact second countable (lcsc) group has a horocone. $G \rtimes G$

For a semisimple real Lie group $G$ the horocone is $G/U$, equivalently $\partial X \times \mathbb{R}$, equipped with a $G$-invariant measure unique up to scaling.

$G = SL(n, \mathbb{R}), \ P = \text{minimal parabolic of } G, \ U \leq P$
Horocone construction

Fix a basepoint $o \in X$. Let $d$ be a $G$-invariant metric on $X$ and $m$ a $G$-invariant measure on $X$. Define the space of “distance-like” functions on $X$ as

$$D := \text{cl}\{x \mapsto d(x, y) + t | y \in X, t \in \mathbb{R}\} \subseteq C(X).$$

We have $G \curvearrowright D$ with $gf(x) := f(g^{-1}x)$.

For $t \in \mathbb{R}$, define $\iota_t : X \to D$ by $\iota_t(x)(y) = d(x, y) - t$, where $y \in X$.

Let $\eta_t := m(B(o, t))^{-1}$ ($t \to \infty \iff \eta_t \to 0$).

Set $\mu_t := \eta_t(\iota_t)_*(m)$. (Goal: $G$-inv. measure on $D$)
Horocone construction, continued

The sequence of measures \( \{\mu_t\}_{t \in \mathbb{R}} \) has a non-zero subsequential weak-\( \ast \) limit \( \mu \) as \( t \to \infty \) whenever \( (X, d) \) has exponential growth.

In particular, such a \( \mu \) exists for any nonamenable lcsc group.

Then \( \mu \) is our desired \( G \)-invariant measure on \( D \), and \( (D, \mu) \) is the horocone for \( G \).
Consider $X = \mathbb{H}^2$ and a boundary point $\xi \in \partial X$.

$\partial X$ can be identified with $G/P$ where $G = SL(2, \mathbb{R})$ and $P$ is the minimal parabolic subgroup of $G$.

Define $\beta_{\gamma}(x) := \lim_{t \to \infty} d(x, \gamma(t)) - d(o, \gamma(t)) \in D$.

The boundary $\partial X = G/P$ is the corresponding equivalence class of Busemann functions. $B_{\gamma_1}(x)$ and $B_{\gamma_2}(x)$ differ by a constant $G$-inv. measure.

Without equivalence, we end up with $\partial X \times \mathbb{R} = G/U$, where $U$ is the maximal unimodular subgroup of $P$. 

Geometric intuition
**Horocones and the IPVT**

The $G$-invariant measure $\mu$ on $D$ determines the Poisson point process on $D$:

The limit $\lim_{t \to 0} \Pi_{\eta_t}$ where each $\Pi_{\eta_t}$ is a Poisson point process on $X$ with intensity $\eta_t$ converges to a Poisson point process on the horocone $G/U$ with positive intensity.

For $x \in X$, if $\beta_{gU} \in G/U$

$$\beta_{gU}(x) = \min\{\beta_{hU}(x) | hU \text{ belongs to the Poisson on } G/U\}$$

then $x$ lives in the IPVT cell of $gU$. 
Cost review
How to prove $G$ and its lattices have fixed price one

Use the following theorems from Abert, Mellick (2021):

The Poisson point process action on $G$ has maximal cost out of all essentially free, measure-preserving actions on $G$.

Let $\Pi$ be a Poisson point process on $G$ and $D$ a complete and separable metric space with a $G$-action. Suppose $\Phi_t(\Pi)$ is a sequence of measurable and equivariant $D$-valued factors of $\Pi$ such that $\Phi_t(\Pi)$ weakly converges to a random process $\Upsilon$ on $D$. Then $\Pi$ and $\Pi \times \Upsilon$ have the same cost.

If $G$ has fixed price one, then so does any lattice in $G$. 
Unbounded walls
Theorem (FMW)

For a higher rank real semisimple Lie group $G$, each pair of cells in its IPVT almost-surely share an unbounded wall.

Sketch of the proof

Let $\Pi$ be the Poisson point process on $G/U$ associated to the IPVT on $X$. Fix any two points belonging to $\Pi$; call them $g_1U, g_2U$. Define $W(r)$ to be set of points $x \in X$ such that:

$$\beta_{g_1U}(x) = \beta_{g_2U}(x)$$

and

$$\beta_{gU}(x) > \beta_{g_1U}(x) + r$$

for every $gU \in \Pi \setminus \{g_1U, g_2U\}$. 

$x \in W(r)$ belongs to $\partial$ only shared by cells of $g_1U, g_2U$.

$x \in W(r)$ only sees cells of $g_1U, g_2U$. 

$\beta_{gU}(x) > \beta_{g_1U}(x) + r$ for every $gU \in \Pi \setminus \{g_1U, g_2U\}$. 

$W(r)$ only sees cells of $g_1U, g_2U$. 

$\beta_{gU}(x) > \beta_{g_1U}(x) + r$ for every $gU \in \Pi \setminus \{g_1U, g_2U\}$. 

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$\beta_{gU}(x) > \beta_{g_1U}(x) + r$ for every $gU \in \Pi \setminus \{g_1U, g_2U\}$.
Sketch of the proof, continued

Define \( W(r) \) to be set of points \( x \in X \) such that:

\[
\beta_{g_1 U}(x) = \beta_{g_2 U}(x)
\]

and

\[
\beta_{gU}(x) > \beta_{g_1 U}(x) + r \quad \text{for every} \quad gU \in \Pi \setminus \{g_1 U, g_2 U\}.
\]

Claim: \( W(r) \) is almost-surely unbounded.

We start with \( x \in X \) such that \( \beta_{g_1 U}(x) = \beta_{g_2 U}(x) \). Then we produce an unbounded set contained in \( W(r) \) from an action on \( x \).
Sketch of the proof, continued

The stabilizer subgroup $S := g_1 U g_1^{-1} \cap g_2 U g_2^{-1}$ fixes $g_1 U, g_2 U$ but mixes up almost every other point of $\Pi$.

$S$ is non-compact only when $G$ is higher rank.

Howe-Moore implies $\lim_{i \to \infty} \mu(B \cap s_i B) = 0$ for Borel $B \subseteq G/U$ and any escaping sequence $\{s_i\}_{i \in \mathbb{N}} \subseteq S$.

Set $B := \{gU \in G/U : \beta_g U(x) < \beta_{g_1} U(x) + r\}$. $\mu(B) < \infty$

As a consequence of the horocone construction, $\mu(B) < \infty$. 

The set of points in $B \cap U$ that are "closer" to $x$ than $B g \cap U x + r$.
Sketch of the proof, continued

By Howe-Moore, there exists a subsequence \( \{s_{i_j}\} \subseteq \{s_i\} \) such that for large enough \( j \ll k \), \( \mu(s_{i_j} B \cap s_{i_k} B) \) is arbitrarily small.

Let \( E_j \) be the event \( \{\Pi(s_{i_j} B) = 0\} \).

We can apply a version of Borel-Cantelli to conclude the \( E_j \) occur infinitely often almost-surely.

For each \( E_j \) which occurs, we have \( s_{i_j}^{-1} x \in W^\perp \). So \( W^\perp \) is unbounded almost-surely.