Goal: Define and understand $\text{cost}(\{\})$

$\text{cost}(\{\}) \leq \text{cost}(\Phi(\{\})))$

$\Phi(\{\}) \cup \text{pmp}$

"Palm meas" $\mathcal{M}_0 = \{\omega \in \mathcal{M} / \omega \in \text{"rooted" config} \}$

Let $\mathcal{R}$ to be OE of $\mathcal{G}/\mathcal{M}$ restricted to $\mathcal{M}_0$

"Rerooting": Declare we $\mathcal{M}_0$ has $w_r \mathcal{R} \rightarrow \mathcal{G}$ given $\omega$.

\{Borel factor graphs $\mathcal{G}_{i:\mathcal{M}} \rightarrow \text{Graph}(\mathcal{M})\} \xrightarrow{\text{a.e.}} \{\text{Borel subsets of } \mathcal{M}_0\}$

Given $G$ directed

Let $\mathcal{A}_G = \{\omega, \eta \in \mathcal{M}_0 \}$

Given $A \subseteq \mathcal{M}_0$, let

$\mathcal{M}_0 = \{\omega, \eta \in \mathcal{M}_0 \}$

The Palm process of $\mathcal{G}$

For $\mathcal{R}(0, \epsilon)$
Given a directed factor graph $G$, let $A_g = \{(\omega, g) \in M_0 \times G \mid \omega \rightarrow g \text{ in } g(\omega)\}$

Given $A \subseteq M_0$, let $G_A(\omega) = \{(g, h) \mid g \omega = h \in A\}$

The Palm process of $\Pi$ in $M_0$ - $\Pi$ conditioned on $M_0$

For $\Pi$ Poisson, $\Pi_0 = \Pi |_{\Omega_0}$

$\Pi_0$ has law $\mu$. Want to define $\Theta_A(\Pi) = \{g \in A \mid g \in \Pi\}$

Let $M_0(A) = \int \Theta_A(\Pi) \, d\mu(\Pi)$

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where $\mu(\Pi) = 1$.

We say $\Pi_0$ is in a Palm region $A$.
\(\mathcal{W}\) has law \(m\). Want to define \(M_0 = \text{Prob}(M_0)\).

Given \(A \subseteq M_0\), define the "\(A\) points" of \(\mathcal{W}\) as

\[\Theta_A(\mathcal{W}) = \left\{ \gamma \in \mathcal{W} \mid \int A \gamma = 1 \right\} \subseteq \mathcal{W}\]

Let \(m_0(A) = \frac{\int A \gamma}{\int \gamma} = \frac{\mathbb{E} [\mathbb{1}_A]}{\mathbb{E} [\mathbb{1}]}\),

where \(\mathbb{1}(W) = 1\).

We say \(\mathcal{W}_0\) is a Palm version of \(\mathcal{W}\) if \(\text{law}(m_0|A) = m_0(A)\).

\[P(\mathcal{W}_0 \subseteq A) = m_0(A)\]
(Holroyd-Percy) The Poisson on $\mathbb{R}^n$ admits $\mathbb{Z}^n \ast w$ as a connected factor graph.

[Toric] Every free ergodic point process on $\mathbb{R}^n$ admits $\text{Vol} \mathbb{Z}^n \ast w$ as a connected factor graph.

$P = \langle S \rangle \ast H$ is $\mathcal{P}$ on $G$.

$G$ admits Cay($H$, $S$) as a connected factor graph.

$\Gamma \subseteq \Gamma(\mathcal{M}_0 \ast \mathcal{M}_0)$ freely generating $\mathbb{R}$. 

$G$ amenable, no $\Gamma$ invariant induced $\mu$ on $C^\infty([\omega])$.
\[ G \text{ amenable}, \text{ no F-invariant mean } M: L^0(G) \to \mathbb{R} \text{.} \]

Induces a mean on \((M_0, \mathcal{M}, \mu_0)\), need means
\[ \mu_0: L^\infty([\omega]) \to \mathbb{R} \]

\(\Sigma\) has law \(\mu_0\).

Given \(A \leq M_0\),
\[ \Theta_A(\Sigma) = \{ \mu \in \mathcal{M} : \mu(A) = 1 \} \]

Let \(M_0(A) = \text{int} \cdot (\Theta_A(\Sigma)) \cap \mathcal{M} \),
\[ \text{We say } \Sigma_0 \text{ in } \{ \Sigma \in \mathcal{M} : M_0(A) = 1 \} \]
\[ M_0(\text{int} \cdot \Sigma_0 + A) = \mathcal{M} \]
The average degree of \( \Gamma \) is factor graph \( \mathcal{G} \):

\[
\text{AvgDeg}(\mathcal{G}) = \frac{1}{E[N_\mu(\Gamma)]} \sum_{e_i \in \mathcal{E}} \text{deg}_e(e_i(\Gamma))
\]

\[
\text{AvgDeg}(\mathcal{G}) = \frac{\text{int}(\Gamma)}{\lambda(\mathcal{G})}
\]

Where \( \lambda(\mathcal{G}) \) is the average degree of \( \mathcal{G} \):

\[
\lambda(\mathcal{G}) = \frac{1}{E[N_\mu(\Gamma)]} \sum_{e_i \in \mathcal{E}} \text{deg}_e(e_i(\Gamma))
\]

\[
\text{Cost}(\Gamma) = \frac{1}{\text{int}(\Gamma)} \sum_{e_i \in \mathcal{E}} f(e_i(\Gamma))
\]

For \( f: \mathcal{G} \rightarrow \mathbb{R} \)
\[ \text{Cost}(\Omega) = 1 = \inf_{g} \left\{ \frac{1}{2} \text{AvgDeg}(g) - \text{intensity}(\Omega) \right\} \]

\[ \text{varint}(\Omega) = 1 \mathbb{E} \left[ N_{\text{in}}(\Omega) \right] \]

\[ \text{int}(\Omega) \int_{\Omega} \mathbb{I}[g \in \Omega] \, d\lambda(g) = \mathbb{E} \left[ \sum_{\Omega \in \Omega} \mathbb{I}[g \in \Omega] \right] \]

\[ \mathbb{E} \left[ \sum_{\Omega \in \Omega} f(g) \right] = \int_{\Omega} \mathbb{E} \left[ \sum_{\Omega \in \Omega} f(g) \right] \, d\lambda(g) \]

for \( f : G \to \mathbb{R} \).
\[ \Phi : \mathbb{M} \to \mathbb{M} \] is a factor map.

\[ \Phi \] can be written as a combination of a thickening and a colouring.

\[ \Theta : \mathbb{M} \to \mathbb{M} \] is a thinning if \( \Theta(w) \subseteq \mathbb{M}_w \).

\[ \Theta : \mathbb{M} \to \mathbb{M} \] is a thickening if \( \Theta(w) \supseteq \mathbb{M}_w \).

\[ \mathbb{C} : \mathbb{M} \to \mathbb{C} \] in a colouring of the underlying set of \( \mathbb{C}(w) \in \mathbb{M}_w \).
in a coloured thickening and can be thinned to $\Phi(\Omega)$.

\[
\text{Cost}(\Omega) = 1 = \inf \left\{ \frac{1}{A(x)} : x \in \Omega \right\}
\]

\[
\text{AW int.}(\Omega) = 1\mathbb{E}\left[ N_u(\Omega) \right]
\]

\[
\int(\Omega) \sum_{q \in \Omega} dA(q) = \mathbb{G}
\]

\[
\mathbb{G} \supset \int(\Omega) \int f(q) d\mathbb{G}(q)
\]

for $f: \mathbb{G} \to \mathbb{R}$. 
\[ \Phi(\Omega) \]

\[ \text{cost}(\Pi) \leq \text{cost}(\Theta(\Pi)) \quad \text{thinning case} \]

\[ \Phi : \mathcal{M} \rightarrow \mathcal{N} \quad \text{for} \]

\[ \text{g}(\Theta(\Pi)) \]

\[ \text{thinned to} \]
case

$\Theta(\Omega)$

$\text{Cost}(\Omega) < \text{Cost}(\Theta(\Omega))$

Thickening case

$\Theta(\Omega)$

$\varepsilon_g(\Theta(\Omega))$

Voronoi of red pts