Higher rank groups have fixed price one
(joint w/ Mikołaj Frączek and Amanda Wilkens)
(also joint work with Miklós Abért)
Theorem [FMW]

Higher rank groups have fixed price one.
(That is, their ess free pmp actions have cost one)

Examples: $SL_3(R)$, $SL_2(R) \times SL_2(R)$, $Aut(T) \times Aut(T)$, $Isom(H^3) \times Aut(T)$
Defn: $\Gamma \subset G$ is a lattice if it is discrete and $\Gamma \backslash G / \Gamma$ has an invariant (Borel) prob. measure.

For $G$ nilpotent, this just means finite index.

Two flavors: if $G / \Gamma$ is compact, we call $\Gamma$ cocompact/uniform.

Forman: Lattices in higher rank simple Lie $G$ are ME Rigid.

If $\Gamma \subset G$ lattice, and $\Delta \sim \text{MRF}$, then $\Delta / \Gamma$ is isom. to a lattice in $G$.

Burger-Mozes: First examples of finitely pres. simple torsion-free groups (special lattices in $\text{Ad}^+T^1 \times \text{Ad}^+T^1$).

$\Gamma \subset \text{Isom}(\mathbb{H}^d)$, torsion-free lattice $\rightarrow$ finite vol. complete hyperbolic $d$-manifold $\mathbb{H}^d$. 

\[ \]
Corollary (Immediate)

Any lattice in a higher rank group has fixed price one known for some lattices, because they have "right angled" generating sets. And they don't always exist.

Defn: The rank of $\Gamma$ is $d(\Gamma)$, min. size of a generating set. Given $\Gamma \leq G$ a lattice, we can write $G = \bigwedge_{\gamma \in \Gamma} H_{\gamma}$ for $\forall \gamma \in \Gamma$. Borel

Defn: $\text{correl}(\Gamma) = \lambda(F)$, where $\lambda$ is Haar measure for $G$.
Corollary (RMW + Abért-M or Carderi)

$G$, higher rank simple Lie group

$\Gamma_n < G$ has $\text{covol}(\Gamma_n) \to \infty$, then the rank gradient

$$\text{RG}(G, \Gamma_n) \leq \lim_{n \to \infty} \frac{d(\Gamma_n) - 1}{\text{covol}(\Gamma_n)}$$
in zero.
$H \leq_{g_i} \Gamma_n$  Then  $d(H) - 1 = \left[ \Gamma_n : H \right] \cdot (n - 1)$

So for any $f_i g$ group $\Gamma$ and $H \leq \Gamma$, we have

$$\frac{d(H) - 1}{[\Gamma : H]} \leq d(\Gamma) - 1$$

**Defn.** A chain of subgroups is $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \ldots$
Theorem (Lachennby + Abért - Jalkin - Zap'rain - Nikolov)

\( \Gamma = \langle S \rangle \) finitely pres., \( \langle \Gamma_n \rangle \) chain of normal f.i. subgroups \( \bigcap_n \Gamma_n = 1 \)

Then at least one of the following is true:

1. for large \( n \), \( \Gamma_n \) is a nontrivial free product
2. \( \text{Cay}(\Gamma/\Gamma_n, S) \) is an expander reg.
3. \( \text{RG}(\Gamma, \langle \Gamma_n \rangle) \) is zero

Moral: positive rank gradient is win-win.
Theorem \cite{Alber-Nikolaev}

If \( \mathcal{S} = \mathcal{S}' \) is a "Farber" chain of subgroups, then \( R(\mathcal{S}, \mathcal{S}') = \cos(\mathcal{S}_0 \mathcal{S}_1) \).

Given freeh. Farber means \( \mathcal{S}_0 \mathcal{S}_1 \) is the coset tree.
How to prove fixed price one (for $G \times Z$ and higher)

- If $G \mathcal{N}(X, \mu) \xrightarrow{f} G \mathcal{N}(Y, \nu)$ in a factor, then
  \[ \text{cost}_a(X, \mu) \leq \text{cost}_a(Y, \nu) \]

- Also true for "weak factor"). Freq. of factors for weakly converging

- Free actions weakly factor onto their "Bernoulli extensions" (e.g. for $G \mathcal{N}(X, \mu)$, \( f \) is $P \times \mathbb{R} \times [0, \pi]$)

- Weak factoring in transitive

- Bern ext. weakly factor onto a "special action"

- Special action has cost one