Perfect matchings a.e. and generically

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Based on joint work with Kun and Sabok and Poulin and Zomback
Theorem (Kőnig)

*Every d-regular bipartite graph has a perfect matching.*
Classical results

Theorem (Kőnig)

Every $d$-regular bipartite graph has a perfect matching.

- A fractional perfect matching is a function $\sigma : E(G) \rightarrow [0, 1]$ such that $\sum_{v \in e} \sigma(e) = 1$ for all $v \in V(G)$. 

   - $\sigma = \frac{d}{d}$ is a fractional perfect matching.
   - Let $F(\sigma) = \sigma - 1(0, 1)$. $F(\sigma)$ has no leaves.
   - We can always fix one edge on a cycle.
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Can we find definable analogs of the above theorem?
Counterexamples

- Irrational rotation graph has no Borel pm a.e./generically.

- (Conley, Jackson, Marks, Seward, Tucker-Drob) There are acyclic $d$-regular Borel graphs with no Borel p.m. for any $d \geq 2$.

- (Kun) There are $d$-regular acyclic pmp Borel graphs with no Borel pm a.e. for all $d \geq 2$. 
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One-ended graphs

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A Borel graph is **one-ended** if each of its connected components is.
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A Borel graph is one-ended if each of its connected components is.

**Theorem (B., Kun, Sabok)**

Every \( d \)-regular hyperfinite one-ended bipartite pmp Borel graph has a Borel pm a.e.

**Theorem (B., Poulin, Zomback)**

Every \( d \)-regular one-ended bipartite Borel graph has a Borel pm generically.
A borel family of sets $\mathcal{T} \subset V(G)^{<\infty}$ is a **toast** if it satisfies properties (1) and (2) of the below definition, and it is a **connected toast** if it also satisfies property 3:

1. $\bigcup_{K \in \mathcal{T}} E(K) = E(G)$,
2. for every pair $K, L \in \mathcal{T}$ either $(N(K) \cup K) \cap L = \emptyset$ or $K \cup N(K) \subseteq L$, or $L \cup N(L) \subseteq K$,
3. for every $K \in \mathcal{T}$ the induced subgraph on $K \setminus \bigcup_{K \not\subseteq L \in \mathcal{T}} L$ is connected.

**Theorem (B., Kun, Sabok)**

Every one-ended hyperfinite Borel graph admits a connected toast a.e.

**Theorem (B., Poulin, Zomback)**

Every one-ended Borel graph admits a connected toast generically.
toasts

Definition
A borel family of sets $\mathcal{T} \subset V(G)^{<\infty}$ is a toast if it satisfies properties (1) and (2) of the below definition, and it is a connected toast if it also satisfies property 3:

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Theorem (B., Kun, Sabok)
Every one-ended hyperfinite Borel graph admits a connected toast a.e.

Theorem (B., Poulin, Zomback)
Every one-ended Borel graph admits a connected toast generically.
Theorem (BPZ)

Any one-ended bipartite $d$-regular Borel graph admits a Borel perfect matching generically.
Theorem (BPZ)

*Any one-ended bipartite \( d \)-regular Borel graph admits a Borel perfect matching generically.*

Let \( \mathcal{T} \) be a connected toast. For every \( L \in \mathcal{T}_1 \) there is an \( m \in \omega \), an \( L \subset K \in \mathcal{T}_{<m} \), and a fractional matching \( \sigma' \) such that

1. \( \sigma'(e) \in \{0, 1\} \) for all \( e \in E(L) \).
2. \( \sigma'(e) = \sigma(e) \) for all \( e \notin E(K) \).
Given a measurable fpm $\sigma$, we can find a measurable $\sigma'$ with $\sigma'(e) \in \{0, \frac{1}{2}, 1\}$ and no cycles.

In fact, any extreme point in the space of measurable fpm has this property.
Lemma

Given a measurable fpm $\sigma$, we can find a measurable $\sigma'$ with $\sigma'(e) \in \{0, \frac{1}{2}, 1\}$ and no cycles.

In fact, any extreme point in the space of measurable fpm has this property.

Let $R$ be the set of such matchings and given $\sigma \in R$ let $L(\sigma) = \sigma^{-1}(\frac{1}{2})$. 
Our strategy: Given $\sigma \in R$, find a $\sigma' \in R$ with $L(\sigma') < L(\sigma)$. 
Improving matchings

Our strategy: Given $\sigma \in R$, find a $\sigma' \in R$ with $L(\sigma') < L(\sigma)$.

It suffices to find $\sigma'$ with

$$\int_{e \in E(G) \setminus L(\sigma)} |\sigma'(e) - \sigma(e)| < \frac{1}{2} \int_{e \in L(\sigma)} |\sigma'(e) - \sigma(e)|.$$
Improving matchings

**Our strategy:** Given $\sigma \in R$, find a $\sigma' \in R$ with $L(\sigma') < L(\sigma)$.

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**Proof.**

We can assume $\sigma' \in R$ by the Choquet–Bishop–de Leeuw theorem and convexity.

Let $A = \{ e \in E(G) \setminus L(\sigma) : \sigma'(e) \neq \sigma(e) \}$ and $B = \{ e \in L(\sigma) : \sigma'(e) \neq \sigma(e) \}$.

We know $L(\sigma') \setminus L(\sigma) \subseteq A$ and $B = L(\sigma) \setminus L(\sigma')$.

Also, $\mu(A) \leq 2 \int_{E(G) \setminus L(\sigma)} |\sigma' - \sigma|$ and $\mu(B) = 2 \int_{L(\chi)} |\sigma' - \frac{1}{2}| = 2 \int_{L(\sigma)} |\sigma' - \sigma|$.

Putting this together gives what we want.
Improving a matching

Let $k$ be large and $\lambda > \varepsilon > 0$ depending on $k$ be tiny. Using a toast find $k$ Borel families of cycles $C_1, \ldots, C_k$ each consisting of pairwise edge-disjoint cycles such that

1. every edge not in $L$ is covered by at most one cycle of $\bigcup_{i=1}^{k} C_i$,  
2. $\mu(\bigcap_{i=1}^{k} E(\bigcup C_i) \cap E(L)) > \frac{1}{2} \mu(L)$,
Improving a matching

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Given this, let $\chi = \frac{\lambda}{d} + (1 - \lambda\sigma)$.

Flip a coin to decide if we’ll add or subtract $\varepsilon$ alternating around each cycle in $\chi$ and let the result be $\sigma'$. 
Improving a matching

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Then $|\sigma'(e) - \sigma(e)| < \lambda + \varepsilon$ for $e \notin L(\sigma)$

and

$$\mathbb{E}|\sigma' - \sigma| = \Omega(\varepsilon \sqrt{k})$$ for $e \in \bigcap_{i=1}^{k} E(\bigcup C_i)$ by Stirling’s approximation.
Problems

- Does every one-ended bipartite Borel graph satisfy \( \chi'_{BM} \leq \Delta(G) \)?

- Does every bipartite \( d \)-regular Borel graph that admits a connected toast have a Borel perfect matching?