On the random order extension property on groups

Andrei Alpeev

Euler Institute, St Petersburg State University

November 8, 2022
Orders on groups

Definition

$X$ a set. An order $\prec$ is a binary relation on $X$ s.t.:

1. $x \prec y$ implies not $y \prec x$;
2. $x \prec y$ and $y \prec z$ implies $x \prec z$.
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Let \( G \) be a countable group. \( G \) acts on \( \text{pOrd}(G) \) :

\[ a(g \prec) b \Leftrightarrow ag \prec bg. \]

this is called R-action (but it is a left \( G \)-action), there is also an \( L \)-action
Invariant (random) orders

**Definition**

A right-invariant order on $G$ is a $G$-invariant point on $\text{pOrd}(G)$. 
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A **group satisfies the invariant order extension property** if every partial invariant order could be extended into a total invariant order.

**Theorem (Rhemtulla-Formanek, early 70's)**

Torsion-free nilpotent groups have the invariant order extension property.

No longer true even for metabelian!
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IRO-extension property

Let be $X$ a set. Denote $\text{OrdExt}(X) \subset p\text{Ord}(X) \times t\text{Ord}(X)$ the set of all pairs $(\omega, \omega')$ s.t. $\omega \in p\text{Ord}(X)$, $\omega' \in t\text{Ord}(X)$ and $\omega \subset \omega'$ ($\omega'$ extends $\omega$).
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A general question: lifting invariant measures over topological extensions: $G \curvearrowright X \rightarrow Y$ Possible for all extension pairs iff $G$ is amenable.
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A general question: lifting invariant measures over topological extensions:

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\downarrow \\
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\end{array}
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Possible for all extension pairs iff $G$ is amenable.
Partial results

Theorem (A. - Meyerovitch - Ryu 20', Stepin? 70's)

*Amenable groups have the IRO extension property.*
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Theorem (Glasner-Lin-Meyerovitch 22')
$SL_3(\mathbb{Z})$ does NOT have the IRO extension property.
Partial results

Theorem (A. - Meyerovitch - Ryu 20’, Stepin? 70’s)

*Amenable groups have the IRO extension property.*

Theorem (Glasner-Lin-Meyerovitch 22’)

$SL_3(\mathbb{Z})$ does *NOT* have the IRO extension property.

Counterexample: semigroup of matrices with non-negative entries generates a partial invariant order, significantly reworked argument by Witte-Morris 94’.
Main result

Theorem

Nonamenable groups do not satisfy the IRO extension property.
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*Nonamenable groups do not satisfy the IRO extension property.*

Thus, amenable $\iff$ IRO extension property.
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Theorem

Nonamenable groups do not satisfy the IRO extension property. Thus, amenable $\Leftrightarrow$ IRO extension property.

Explicit set of counterexamples for the lifting problem:

$$G \curvearrowright X \quad \Downarrow \quad G \curvearrowright Y$$
Idea of proof

If $G' < G$ and $G$ has the IRO extension property then so does $G'$. \

maybe $F_2$ has no IRO extension property?
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- each non-amenable group contains $F_2$
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if $G' < G$ and $G$ has the IRO extension property then so does $G'$. Idea:

▶ maybe $F_2$ has no IRO extension property?
▶ each non-amenable group contains $F_2$ [Olshanski, early 80’s].
Equivalence relations

Definition

\((X, \mu)\) a standard probability space. \(E\) is a countable Borel equivalence relation:

- \(E\) is a Borel subset of \(X \times X\);
- \(E\) is an equivalence relation;
- equivalence classes of \(E\) are at most countable.
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Main example - orbit equivalence relations of measure-preserving actions of countable group on a standard probability space:

$$xEy \text{ iff } y = gx \text{ for some } g \in G.$$
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Equivalence relations are high-level analogs of groups.
Theorem (Gaboriau-Lyons 09’)

Let $G$ be a non-amenable group. There is an essentially free pmp action of $G$ with orbit equivalence relation $E_2$ and an essentially free pmp action of $F_2$ on the same standard probability space with orbit equivalence relation $E_1$ s.t. $E_1 \subset E_2$. 
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Some applications:

- Dixmier problem for lamplighters over non-amenable groups [Monod-Ozawa 09’] ;
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- Dixmier problem for lamplighters over non-amenable groups [Monod-Ozawa 09’] ;
- Ulam non-stability for lamplighters over non-amenable groups [A.22’].
IROs on equivalence relations

$M(X)$ - the space of all prob. measures on $X$. 
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**Definition**

Let \( E \) be a measure preserving Borel equivalence relation on a standard probability space \( (X, \mu) \). An IRO on \( E \) is a map \( f \) s.t.

1. \( f(x) \in M(pOrd([x])_E) \) for all \( x \in X \);
2. \( f(x) = f(y) \) for a.e. \( x \in X \) and all \( y \in x^E \);
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\( E \) has the IRO extension property if for every IRO \( f \) there is a map \( t \) s.t.

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Lemma

Let $E_1 \subseteq E_2$ be two equivalence relations. If $E_2$ has the IRO extension property then $E_1$ has the IRO extension property.

Proof.
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Apply the extension property for $f_2$ and get $t_2$. 
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Apply the extension property for \( f_2 \) and get \( t_2 \).

Restrict \( t_2(x) \) to \([x]_{E_1}\) for each \( x \) to get \( t \) for \( f \).
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Let $\nu$ be a measure on $\text{pOrd}(G)$. 

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For each $x$, we identify $G$ with $[x]_E$. 

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For each $x$, we identify $G$ with $[x]_E$ (and so $M(\text{pOrd}(G))$ with $M(\text{pOrd}([x]_E)))$. So we get an IRO $f$. Apply the extension property for $E$ to $f$. 


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Apply the extension property for $E$ to $f$.
get an invariant measure on $X \times \text{pOrd}(G) \times \text{tOrd}(G)$. 
\qed
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IRO extension property for $G$ implies that for $E$. 

proof continued
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Idea:
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Idea:

- IRO on $E$ gives a joining of $G \bowtie (X, \mu)$ with $G \bowtie \text{pOrd}(G)$.
- project to $\text{pOrd}(G)$. 
proof continued

Proof.
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▶ project to $\text{pOrd}(G)$.
▶ apply the extension property for $G$
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- Apply the extension property for $G$.
- Relatively independent toining of $G \bowtie X \times \text{pOrd}(G)$ and $G \bowtie \text{pOrd}(G) \times \text{tOrd}(G)$ over the common factor $\text{pOrd}(G)$. 
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- Decompose over $X$. 

□
Counterexample for $F_2$

Why there is a counterexample for $F_2$?
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Why there is a counterexample for $F_2$?

$\pi : F_2 \rightarrow SL_3(\mathbb{Z})$, lift over projection.
For $a, b \in G$ denote:

\[ sml^+ \sqsubseteq (a, b) = \{ \prec \in \text{Ext}(\sqsubseteq) \mid \exists q > 0 \forall n > 0 a - q b \prec e \} \]

\[ sml^- \sqsubseteq (a, b) = \{ \prec \in \text{Ext}(\sqsubseteq) \mid \exists q > 0 \forall n > 0 e \prec b - n a q \} \]

\[
\begin{align*}
a_1 &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
a_2 &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
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a_4 &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
a_5 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
a_6 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}
\end{align*}
\]
For $a, b \in G$ denote:

$$sml^+_E(a, b) = \{ \preceq \in \text{Ext}(E) \mid \exists q > 0 \forall n > 0 \ a^{-q}b^n \prec e \}$$
$$sml^-_E(a, b) = \{ \preceq \in \text{Ext}(E) \mid \exists q > 0 \forall n > 0 \ e \prec b^{-n}a^q \}$$
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Denote \( s^m_\ell \subseteq T_{\ell=1}^{6} \) and \( s^m_\ell \subseteq (a\ell, a\ell-1) \) and \( s^m_\ell \subseteq T_{\ell=1}^{6} \) and \( s^m_\ell \subseteq (a\ell, a\ell+1) \).

**Lemma (GLM22)**

\[ \text{Ext}(\subseteq) = s^m_\ell \cup s^m_\ell \]

Let \( F \) be a free group and let \( \pi: F \to \Gamma \) be an epimorphism. A **transversal** is any map \( \phi \) from \( \Gamma \) to \( F \) such that \( \pi \circ \phi \) is the identity map on \( \Gamma \).

Fix any \( \alpha_1, \ldots, \alpha_6 \in F \) such that \( \pi(\alpha_i) = a_i \). Define \( \phi(a_n^i a_m^i+1) = \alpha_n^i \alpha_m^i+1 \), for \( i = 1, \ldots, 6 \mod 6 \), and \( n, m \in \mathbb{Z} \); we define \( \phi \) on remaining elements of \( \Gamma \) arbitrarily to get a transversal.
Denote \( \text{sml}^- = \bigcap_{i=1}^6 \text{sml}^-(a_i, a_{i-1}) \) and \( \text{sml}^+ = \bigcap_{i=1}^6 \text{sml}^+(a_i, a_{i+1}) \).
Denote \( \text{sml}_- = \bigcap_{i=1}^{6} \text{sml}_- (a_i, a_{i-1}) \) and 
\( \text{sml}_+ = \bigcap_{i=1}^{6} \text{sml}_+ (a_i, a_{i+1}) \).

**Lemma (GLM22)**

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Let \( F \) be a free group and let \( \pi : F \to \Gamma \) be an epimorphism. A transversal is any map \( \varphi \) from \( \Gamma \) to \( F \) such that \( \pi \circ \varphi \) is the identity map on \( \Gamma \).

Fix any \( \alpha_1, \ldots, \alpha_6 \in F \) such that \( \pi(\alpha_i) = a_i \). Define 
\( \varphi(a_i^n a_{i+1}^m) = \alpha_i^n \alpha_{i+1}^m \), for \( i = 1, \ldots, 6 \mod 6 \), and \( n, m \in \mathbb{Z} \); we define \( \varphi \) on remaining elements of \( \Gamma \) arbitrarily to get a transversal.
Thanks!