Topology versus Borel structure for actions

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McGill DDC seminar, October 18, 2022
Background

For a topological space $X$, $\mathcal{O}(X) := \{\text{open sets in } X\}$.
For a Borel space $X$, $\mathcal{B}(X) := \{\text{Borel sets in } X\}$. 
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Theorem (classical)

*For a “nice” Borel space $X$, every $B \in \mathcal{B}(X)$ is in some “nice” compatible topology. Moreover, countably many $B_i \in \mathcal{B}(X)$ may be made open at once.*

“Nice” Borel space = standard Borel space
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Theorem (Pettis)

For a Polish group $G$, $U \in \mathcal{B}(G)$ is a nbhd of 1 iff $\exists$ ctbl $G = \bigcup_i B_i$, $B_i B_i^{-1} \subseteq U$. 
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$\implies U \supseteq B_iB_i^{-1} \supseteq V_iV_i^{-1} \ni 1$. □
Group actions

\[ G \curvearrowright X: \]

\[ G \cdot x \cong G/G_x \]

\[ G \cdot y \]

\[ G \cdot z \]

Theorem (Becker–Kechris 1996)

Let \( G \) be a Polish group, \( X \) be a standard Borel \( G \)-space. For \( B \in B(X) \), TFAE:

(i) \( B \) is potentially open in some compatible Polish top making \( G \rtimes X \) cts;

(ii) \( B \) is orbitwise open: for each \( x \in X \), \( B \) is open in quotient top on \( G \rightarrow G \cdot x \).

Moreover, ctbly many orbitwise open \( B_i \) may be made open at once.

Corollary

Every standard Borel \( G \)-space may be made into a Polish \( G \)-space.

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Better understanding of Becker–Kechris, and topological realizations in general.
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1. Detailed characterization of potentially open Borel sets.
   (i) potentially open
   (ii) orbitwise open
   (iii) preimage under action is ctbl union of Borel rectangles
   (iv) translates are ctbly generated under unions
   (v) ctbl union of Vaught transforms

   ▶ more “topological” than original proof (and proof of Hjorth)
   ▶ does not use strong Choquet game
   ▶ easily generalizes to other contexts

3. Extend to various other contexts.
   ▶ potentially open $n$-ary relations
   ▶ non-Hausdorff (quasi-Polish) $G$-spaces
   ▶ groupoid actions
   ▶ actions preserving existing topology
   ▶ non-second-countable actions (on point-free “spaces”)
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Quasi-Polish spaces

Definition (de Brecht 2013) A **quasi-Polish space** is a “non-Hausdorff Polish space”.

- **Fact** Polish = quasi-Polish + $T^3$.
- **Fact** Change of topology works the same for quasi-Polish as for Polish spaces.
- **Fact** Quasi-Polish spaces are standard Borel and (completely) Baire.
- **Fact** Quasi-Polish group = Polish group.
Quasi-Polish spaces

Definition (de Brecht 2013) A quasi-Polish space is a “non-Hausdorff Polish space”.

- second-countable, completely quasi-metrizable
- $\Pi^0_2$ subspace of $\mathcal{S}^\mathbb{N}$, where $\mathcal{S} = \{0, 1\}$ w/ $\{1\}$ open, and $\Pi^0_2$ means $\bigcap_i (U_i \Rightarrow V_i)$
- continuous open $T_0$ quotient of $\mathbb{N}^\mathbb{N}$
  - recall: Polish = continuous open $T_3$ quotient of $\mathbb{N}^\mathbb{N}$
- $T_0$ quotient of a Polish group action on Polish space (topol ergodic decomp)
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**Definition (de Brecht 2013)** A **quasi-Polish space** is a “non-Hausdorff Polish space”.

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Let $G$ be a Polish group, $\mu : G \times G \to G$ group mult, $\alpha : G \times X \to X$ Borel action.
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Transfer top on $G$ to each $\alpha^{-1}(x)$. 

\[
\begin{aligned}
\{(g,y) | gy = x\} &= \alpha^{-1}(x) \\
G \times X &\xrightarrow{\alpha} X \\
&\xrightarrow{\alpha^{-1}} \bigcup V W \subseteq U \times (V \times (W \star B)) \\
\end{aligned}
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Vaught transforms

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**Definition** For $A \in \mathcal{B}(G \times X)$,

$$\exists^*_\alpha(A) := \{x \in X \mid A \text{ nonmgr in } \alpha^{-1}(x)\}.$$
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(aka: $B^{\Delta U^{-1}}$) $U \ast B := \exists^*_\alpha(U \times B)$. 

**Diagram**
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Definition For $U \in \mathcal{B}(G)$ and $B \in \mathcal{B}(X)$,

$$(\text{aka: } B^\Delta U^{-1}) \ U*B := \exists^*_\alpha(U \times B).$$

$$U \in \mathcal{O}(G) \implies \alpha^{-1}(U*B) = \bigcup_{VW \subseteq U} (V \times (W*B)).$$
Core Theorem (C. 2022)

Let $G$ be a Polish group, $X$ be a quasi-Polish space with a Borel $G$-action s.t.

$$\mathcal{O}(G) \ast \mathcal{O}(X) \subseteq \mathcal{O}(X).$$

Then $\langle \mathcal{O}(G) \ast \mathcal{O}(X) \rangle$ is a compat quasi-Polish top making action cts.
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$$\mathcal{O}(G \times X) \xleftarrow{\exists^*_\alpha} \langle \mathcal{O}(G) \ast \mathcal{O}(X) \rangle \subseteq \mathcal{O}(X)$$
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\[ \mathcal{O}(G \times X) \xrightarrow{\exists^*_\alpha} \langle \mathcal{O}(G) \ast \mathcal{O}(X) \rangle \subseteq \mathcal{O}(X) \]

Lemma Let $f : X \twoheadrightarrow Y$ be a Borel surj from a q-Pol sp to a st Borel sp. Suppose each fiber $f^{-1}(y)$ is equipped with a coarser q-Pol top “in a Borel way”, and $f$ is cts wrt $\exists^*_f(\mathcal{O}(X))$. Then $Z :=$ smallest fbwise closed (in finer top) comgr (in coarser top) $\subseteq X$ is $\Pi^0_2$, and $f|_Z : Z \twoheadrightarrow Y$ is cts open $T_0$ quotient with $\exists^*_f = \exists^*_f|_Z$. \qed
Theorem (Becker–Kechris; C.)

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(iv) $\{gB \mid g \in G\} \subseteq$ closure under $\bigcup$ of ctbly many $B_i \in \mathcal{B}(X)$;
(v) $B = \bigcup_i (U_i * B_i)$ for ctbly many $U_i \in \mathcal{O}(G)$ (or $\mathcal{B}(G)$), $B_i \in \mathcal{B}(X)$.

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Proof. (i) $\implies$ (ii),(iv)
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Proof. (i) $\implies$ (ii),(iv) $\implies$ (iii) by

Theorem (Kunugui–Novikov) Let $f : X \to Y$ be a Borel map between st Borel spaces, $S \subseteq \mathcal{B}(X)$ be ctble. If $A \in \mathcal{B}(X)$ is $f$-fiberwise a union of sets in $S$, then

$$A = \bigcup_{S \in S} (f^{-1}(B_S) \cap S) \quad \text{for } B_S \in \mathcal{B}(Y).$$
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**Proof.** (i) $\implies$ (ii),(iv) $\implies$ (iii) $\implies$ (v) by $B = \exists^*_\alpha(\alpha^{-1}(B))$
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Moreover, ctbly many such $B$ may be made open at once.

**Proof.** (i) $\Rightarrow$ (ii),(iv) $\Rightarrow$ (iii) $\Rightarrow$ (v) (both versions equiv by Pettis).
Topological realization

Theorem (Becker–Kechris; C.)

Let $G$ be a Polish group, $X$ be a standard Borel $G$-space. For $B \in \mathcal{B}(X)$, TFAE:

(i) $B$ is potentially open in some compat (quasi-)Polish top making $G \bowtie X$ cts;
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(v) $\implies$ (i): To make ctbly may $U_i \ast B_i$ open, find compat q-Pol top $\mathcal{O}(X)$ containing each $B_i$ and closed under $\mathcal{O}(G)^*$. 
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(v) $\implies$ (i): To make ctbly may $U_i \ast B_i$ open, find compat q-Pol top $\mathcal{O}(X)$ containing each $B_i$ and closed under $\mathcal{O}(G)^\ast$. By Core Thm, $\langle \mathcal{O}(G) \ast \mathcal{O}(X) \rangle$ works. \qed
Comparison with original proof

Core Theorem (Becker–Kechris 1996)

Let $G$ be a Polish group, $X$ be a zero-dimensional Polish with a Borel $G$-action, $\mathcal{U} \subseteq \mathcal{O}(G)$ and $\mathcal{A} \subseteq \mathcal{O}(X)$ be countable bases s.t. $\mathcal{A}$ is a Boolean algebra and $\mathcal{U} \ast \mathcal{A} \subseteq \mathcal{A}$.

Then $\langle \mathcal{U} \ast \mathcal{A} \rangle$ is a compat Polish top making action cts.

The proof consists of showing:

1. the action is cts;
2. the topology is $T_3$;
3. the topology is strong Choquet.

Combining 1. and 2. with our Core Thm yields a Polish top realization.
Comparison with Hjorth–Sami

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### Better Core Theorem (C.)

Let $G$ be a Polish group, $X$ be a quasi-Polish with a Borel $G$-action, $\mathcal{U} \subseteq \mathcal{O}(G)$ and $\mathcal{A} \subseteq \mathcal{O}(X)$ be countable bases s.t. $\mathcal{U} = \mathcal{U}^{-1}$, $\mathcal{A}$ is a **lattice**, and $\mathcal{U} \ast \mathcal{A} \subseteq \mathcal{A}$.

Then letting $\mathcal{B}$ be the Boolean algebra generated by $\mathcal{A}$, $\langle \mathcal{U} \ast \mathcal{B} \rangle$ is a compat Polish top making the action cts, and is 0-d if $\mathcal{U}$ consists of cosets.
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**Corollary** For a quasi-Polish $G$-space $X$ and $B \in \Sigma^0_\xi(X)$, $\xi \geq 2$, $\mathcal{U} \ast B$ is open in a finer **Polish** topology $\subseteq \Sigma^0_\xi(X)$ (0-d if $G$ is non-Archimedean).
Automatic continuity for actions

Theorem (classical for Polish; C.)

Let $G$ be a Polish group, $X$ be a quasi-Polish $G$-space with a Borel action of $G$ via homeomorphisms. Then the action is jointly continuous.

In other words, “if the action preserves an existing topology, we may find a topological realization compatible with that existing topology”. 
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So $\mathcal{O}(G) \ast \mathcal{O}(X) \subseteq \mathcal{O}(X)$. By Pettis, $B = \bigcup_i (U_i \ast B_i) \in \langle \mathcal{O}(G) \ast \mathcal{O}(X) \rangle = \mathcal{O}(X)$. □
Definition A **groupoid** \( G \) consists of two maps \( G \xrightarrow{\sigma} G_0 \) (src, tgt) and operations

![Diagram of groupoid](image-url)
Groupoids

Definition A groupoid $G$ consists of two maps $G \xrightarrow{\sigma} G_0$ (src, tgt) and operations

An action of $G$ on a bundle $p : X \to G_0$ is a map $\alpha : G \times_{G_0} X \to X$ s.t. each $g : x \to y \in G$ acts via a map $p^{-1}(x) \to p^{-1}(y)$.
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\[ g^{-1} \]

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**Note** Most open quasi-Polish groupoids are not Polish!
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Let $G$ be an open q-Pol gpd, $p : X \to G_0$ a st Borel $G$-space. For $B \in \mathcal{B}(X)$, TFAE:

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Let $G$ be an open q-Pol gpd, $p : X \rightarrow G_0$ a standard Borel bundle of q-Pol spaces with a $G$-action via homeos. Then $\exists$ global q-Pol top on $X$ restricting to fiberwise tops.
Open relations

For a group(oid) action on $X$, we know $B \in \mathcal{B}(X)$ potentially open iff orbitwise open. What about $R \in \mathcal{B}(X^n)$?
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Similarly for groupoids, multi-sorted structures, change of topology, ...