Question 1: Let α, β be free, ergodic, measure preserving actions $\Gamma_n \overset{\sigma}{\to} (X, \mu)$ and $\Gamma_m \overset{\sigma}{\to} (X, \mu)$. If α, β produce the same orbits, must $n = m$?

Gaboriau 1998 \quad \text{Yes!}

Question 2: What about $\mathbb{Z} \times \mathbb{Z}$?

Theorem (Dye '59 for $\mathbb{Z}$, Ornstein–Weiss '80 for amenable) Any two free ergodic, m.p. actions of amenable groups are orbit equivalent.

\Rightarrow \text{answer to Q2 is no!}

- $E$ a countable Borel equivalence relation on $(X, \mu)$
- $E$ is probability measure preserving if for any Borel automorphism $\tau$ that permutes the $E$-classes, $\tau$ is pmp $\Rightarrow \mathbb{E} (A) = \mu (A)$
- $G$ is a graphing of $E$ if $E = E_G$
- $T$ is a treeing of $E$ if $T$ is an acyclic graphing.
Cost = "\# edges"

For a loc. countable pmp \( G \)

\[
\text{Cost}_\mu(G) := \frac{1}{2} \int |G_x| \, d\mu \\
= \int |G_x| \, d\mu
\]

\[
\text{Cost}_\mu(E) := \inf \{ \text{Cost}_\mu(G) : G \text{ is a Borel graphing} \}
\]

**Theorem (Gabrielli '98)**
Treeings achieve cost. (\( T \) Borel treeing of \( E \) \( \Rightarrow \) \( \text{Cost}_\mu(E) = \text{Cost}_\mu(T) \))

(a) \( T \) has bounded degree (\( \leq d \))
(b) \( \exists L, M > 0 \) s.t.
\[
\frac{1}{M} d_T \leq d_G \leq L d_T
\]

Euler "\# edges - \# vertices"

\[
\text{Euler}_\mu(G) := \text{Cost}_\mu(G) - M(X)
\]
**Proof** Fix a graphing $G$ of $E$. Want to show $\text{cost}_\mu(G) \geq \text{cost}_\mu(T)$.

Assume that:

1. $T$ has bounded degree $(\leq d)$
2. $\exists L, M > 0$ s.t.
   \[
   \frac{1}{M} d_T \leq d_G \leq L d_T
   \]

Idea 1) "Implement" $G$ via edges of $T$

$$X' := X \cup \{ (x, e) : x \text{ lies on the interior of the } T\text{-path connecting } e \}$$

$G' :=$ corresponding $T$-edges

Euler "# edges - # vertices"

$$\text{Euler}_\mu(G) := \text{cost}_\mu(G) - M(X)$$

$$M'(A) := \int \{ \text{proj}^{-1}(x) \cap A \} \, d\mu_X$$

* $M'$ is still finite: $\text{proj}(y) = X \Rightarrow y \in (x, e)$
\( \leq d^{2M} \) possibilities for endpoints

\[ e \in G \]

\( m' \big| x = m \)

\( \text{Euler}_{m'}(G') = \text{Euler}_{m}(G) \)

\( \text{cost}_{m'}(G') - m'(X') = \text{cost}_{m}(G) - m(X) \)

\( \text{cost}_{m'}(G') = \text{cost}_{m}(G) + m'(X' \setminus X) \)

Fix a Borel directing of \( G, \overrightarrow{G} \) and direct \( G' \) accordingly

(\( \Rightarrow \) each fake vertex has out\( \deg = 1 \) & the out\( \deg \) of each true vertex is the same)

\[ \text{cost}_{m'}(G') = \sum_{x, \overrightarrow{G}} | \overrightarrow{G}_x | \, dm' \]

\[ = \sum_{x, \overrightarrow{G}} | \overrightarrow{G}_x | \, dm' + \sum_{x', \overrightarrow{G}} | \overrightarrow{G}_x | \, dm' \]

\[ = \text{cost}_{m}(G) + m'(X' \setminus X) \]
Idea 2: If $(x,e)$ is a fake vertex, look at true vertex $(x,\emptyset)$:

1. There is a path from $x$ to $x$.

2. This path must have a backtracking.

3. Thus $z \rightarrow w \rightarrow z$.

4. "Folding" $(z,f)$.

5. Countably color triples of points to be folded (so that if two triples are distance $\leq d^{2m}$, get different colors).

6. $Y_0 := X'$, $H_0 := G'$.

7. $Y_{m+1}, H_{m+1}$ by folding triples of color $m$, choosing which vertex survives so that true vertices always survive.
(x, e) \rightarrow (x', e)

\text{y} \uparrow \rightarrow y

\begin{align*}
d_{G'}((x, e), (y, \emptyset)) & \leq M \\
d_T(x, y) & \leq M \\
d_G(x, y) & \leq LM \\
d_G(x, y) & \leq LM^2 \\
d_{G'}((x, e), (x, \emptyset)) & \leq M + LM^2
\end{align*}

After iterating \((X_n, G_n) \leq M + LM^2\)
many times, \(X_{M+2LM^2} = X\)

\(G_{M+LM^2} = T\)

\(Euler_{m+1}(G) = Euler_{m+1}(G') \geq Euler_{m+1}(G_{M+LM^2}) = Euler_{m}(T)\)
\[ \text{Cost}_m(G) - m(X) \geq \text{Cost}_n(T) - n(X) \]