Borel Determinacy in 50 ($+\varepsilon$) Minutes

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Gale-Stewart games; infinite two player games of perfect information where the players, denoted I and II, alternate moves.
Games

Given a nonempty set of moves \( M \) and a payoff set \( A \subseteq M^\mathbb{N} \), we define the game \( G(A) \):

- A **position** is a finite sequence \( p \in M^{<\mathbb{N}} \)
- A **run** is an infinite sequence \( (x_n)_{n \in \mathbb{N}} \in M^\mathbb{N} \)
- Players I and II take turns playing **moves** \( x \in M \)
- Player I wins iff the run \( (x_n)_{n \in \mathbb{N}} \in A \)
In practice we want to play a **game with rules**:

- Restrict moves to a subtree (without leaves), say $T$
- Equivalent to games without rules, up to changing the payoff set

$G(A; T)$:
**Games: Determinacy**

**Definition**

A game $G(A; T)$ is **determined** if one of the players has a winning strategy.

A **strategy** for player I is a function $\sigma : T \to M$ that tells the player what move to play at any even position $p \in T$ and is **winning** for I if every run consistent with $\sigma$ is in $A$.

A strategy $\tau$ for II is defined analogously.

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*Remark:* Bounded games are determined.
We equip $[T]$ with the topology whose basic open sets are $[T_p]$, $p \in T$, where

$$T_p := \{ q \in T : (q \subseteq p) \lor (p \subseteq q) \}$$

denotes the game subtree at position $p$. 
Theorem (Clopen Determinacy)

If $A \subseteq [T]$ is clopen, then $G(A; T)$ is determined.

Proof.

Suppose II doesn't have a winning strategy. Call a node "heavy" if II doesn't have a winning strategy from that point. Chase the heaviness into $A$. 

\[ \square \]
### Previous Determinacy Results

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<th>Theorem (Gale, Stewart (1953))</th>
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<td>Open/Closed sets (i.e. $\Pi^0_1$) are determined.</td>
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<th>Theorem (Wolfe (1955), Davis (1964), Paris (1972))</th>
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<td>$\Sigma^0_2$, resp. $\Pi^0_3$, resp. $\Sigma^0_4$ sets are determined.</td>
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<th>Theorem (Borel Determinacy; Martin (1975))</th>
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<td>All Borel sets are determined.</td>
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Motivation: Why do we care?

It turns out that regularity properties of subsets of a Polish space are naturally deduced from the determinacy of infinite games, including:

- measurability
- Baire measurability
- the perfect set property

Borel determinacy tells us that Borel sets are the "nicest" possible.
Let $X$ be a perfect Polish space.

**Definition**

A set $B \subseteq X$ has the **perfect set property** (PSP) if it’s either countable or contains a Cantor set ($2^\mathbb{N}$).

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\begin{align*}
\text{Let } X & \subseteq \mathbb{R}^2 \\
\text{\textbf{Definition}} & \\
\text{A set } B & \subseteq X \text{ has the perfect set property (PSP) if it's either countable or contains a Cantor set } (2^\mathbb{N}).
\end{align*}
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Proof of Borel Determinacy

... but first a slight detour into taboos

\[ \text{Closed} \Rightarrow \text{open} \]
**Definition (Game tree with taboos)**

A **game tree with taboos** is a triple \( T := \langle T, T_I, T_{II} \rangle \) where

- \( T \) is game tree, but with leaves
- \( T_I \) is the set of taboos for player I
- \( T_{II} \) is the set of taboos for player II
For a game with taboos $G(A; T)$, we still consider only the space of infinite branches equipped with the topology as before.

Games with taboos can be modeled as infinite games without taboos:

Remark: This may change the Borel complexity of subsets of $[T]$.
Determinacy of games with taboos

Lemma

*Clopen games with taboos are determined.*

Note: Clopen determinacy for games without taboos does not give us this result for free! (but the proof is similar in spirit)
Given a Borel game $G(A; T)$ we want to build an auxiliary clopen game $G(\tilde{A}, \tilde{T})$ s.t. winning strategies in $G(\tilde{A}, \tilde{T})$ map to winning strategies in $G(A; T)$. 
Game coverings

Definition (Covering)
A covering of a game tree $T$ is a tuple $\langle \tilde{T}, \pi, \phi \rangle$, of a game tree $\tilde{T}$, a position map $\pi : \tilde{T} \Rightarrow T$, and a strategy map $\phi : \tilde{T} \xrightarrow{S} T$, such that:

Lemma
For all $A \subseteq [T]$, if $\tilde{\sigma}$ is a winning strategy for $G(\pi^{-1}(A); \tilde{T})$, then $\sigma := \phi(\tilde{\sigma})$ is a winning strategy for $G(A; T)$. 

\[ C : \tilde{T} \xrightarrow{S} T \]
**Definition (Unraveling)**

Given a set $A \subseteq [T]$, we say a covering $\langle \tilde{T}, \pi, \phi \rangle$ of $T$ unravels $A$ if $\pi^{-1}(A)$ is clopen in $[\tilde{T}]$.

**Corollary**

If there is a covering of $T$ that unravels $A \subseteq [T]$, then $G(A; T)$ is determined.

Our goal: Unravel every Borel set $A$.
Proof by Induction

Recall each Borel set is obtained from open (or closed) sets by applying complements and ctbl unions.

Inductive proof:

- Base case: unravel closed sets
- Complements: $A$ unravelled $\implies A^c$ unravelled
- Ctbl $\bigcup$: each $A_n$ unravelled $\implies \bigcup_{n \in \omega} A_n$ unravelled
$A = \bigcup_{n \in \omega} A_n$ unravels if $A_n$’s unravel.

The preimage $A$ in $T_\infty$ is clopen.
A word on the base case
Questions?
Thank you!

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