Introduction to Poisson processes

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Abstract
The Poisson process provides a canonical way to build a probability space from a possibly infinite measure space. We give an introduction to Poisson processes from a descriptive set theorist’s perspective, with some applications to constructing free pmp actions of Polish groups.

1 Idea

Input: A measure space \((X, \rho)\), usually with \(X\) standard Borel and \(\rho\) \(\sigma\)-finite.
Output: A probability space \((N(X), \text{Pois}_X(\rho))\) canonically and faithfully built from \((X, \rho)\).

Example. For \((X, \rho) = (\mathbb{R}, \lambda)\), a typical sample from \text{Pois}_\mathbb{R}(\lambda)\) is a countable subset of \(\mathbb{R}\):

\[
\begin{array}{cccccccccc}
\vdots
& \vdots
& \vdots
& \vdots
& \vdots
& \vdots
& \vdots
& \vdots
& \vdots
& \vdots
\end{array}
\]

On average, there are \(\lambda([n, n+1]) = 1\) points per unit interval.

Every locally compact Polish group \(G\) admits a free pmp Borel action.

Proof. If \(G\) is compact, take Haar measure \(\mu\).
Otherwise, take \(\text{Pois}_G(\mu)\).

Remark. If \(G\) is countable, we can also take \((2^G, \mu^G)\) (or \((X^G, \mu^G)\)) with shift action.
In this case, \(\text{Pois}_G(\rho) = (N^G, \text{Pois}_1(\rho(1))^G)\) (see later Example).

If \(X\) is a locally compact Polish metric space, then \(\text{Iso}(X)\) admits a free pmp action.
In particular, \(\text{Iso}(X) \leq \text{Aut}(\mu)\).

Proof. By [Loomis 1949], \(X\) admits a regular \(\sigma\)-finite \(\text{Iso}(X)\)-invariant measure \(\mu\).
Take \(\text{Pois}_X(\mu)^N\).

2 Spaces of measures

Definition. Let \(X\) be a Borel (i.e., measurable) space.
\[
\mathcal{M}(X) := \{\mu \in [0, \infty]^{\mathcal{B}(X)} \mid \mu \text{ is a measure}\}, \text{ w/ induced Borel structure}
\]
\[
\mathcal{N}(X) := \{\nu \in \mathcal{M}(X) \mid \nu \text{ is } \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}-valued\}
\]
\[
\mathcal{M}_{<\infty}(X) := \{\mu \in \mathcal{M}(X) \mid \mu(X) < \infty\}
\]
\[
\mathcal{M}_1(X) := \{\mu \in \mathcal{M}(X) \mid \mu(X) = 1\}
\]
If \(X\) is standard Borel, then so is \(\mathcal{M}_1(X)\), as is \(\mathcal{M}_{\leq 1}(X)\), hence also \(\mathcal{M}_{<\infty}(X) = \bigcup_n \mathcal{M}_{\leq n}(X)\).
Example. A countable subset $N \subseteq X$ can be identified with $\nu \in \mathcal{N}(X)$ given by $\nu(B) := |N \cap B|$. In general, we can think of $\nu \in \mathcal{N}(X)$ as a “multiset”, where $\nu(B)$ is the number of points in “$\nu \cap B$”; the same point may occur multiple times. (Note that there are weird measures in $\mathcal{N}(X)$ which are not a countable sum of Dirac deltas, e.g., $\nu(B) := \infty$ if $B$ is uncountable, else $\nu(B) = 0$. We could restrict to the subspace of countably supported $\nu \in \mathcal{N}(X)$; but below we will restrict to a further subspace $\mathcal{N}_{\prec \infty}(X)$ which is moreover standard Borel. For maximum generality, we will begin by working on the full non-standard Borel space $\mathcal{N}(X)$.)

3 Poisson measures

The Poisson process construction takes $\mathcal{M}(X) \ni \rho \mapsto \text{Pois}_X(\rho) \in \mathcal{M}_1(\mathcal{N}(X))$.

For $X = 1$. Given $\rho \in \mathcal{M}(1) \cong [0, \infty]$, we define $\text{Pois}(\rho) = \text{Pois}_1(\rho) \in \mathcal{M}_1(\mathcal{N}(1)) \cong \mathcal{M}_1(\mathbb{N})$ by

$$\text{Pois}(\rho)(n) := e^{-\rho n^n} \frac{\rho^n}{n!}$$

for $\rho < \infty$ and $n \in \mathbb{N}$ (with $\text{Pois}(\rho)(\infty) = 0$).

For $\rho = \infty$, by convention $\text{Pois}(\infty) := \delta_{\infty} \in \mathcal{M}_1(\mathbb{N})$.

**Proposition.** $\text{Pois}(\sigma + \tau) = \text{Pois}(\sigma) * \text{Pois}(\tau) := +_{\ast}(\text{Pois}(\sigma) \times \text{Pois}(\tau))$, where $+ : \mathbb{N}^2 \to \mathbb{N}$.

**Proof.** If $\sigma = \infty$ or $\tau = \infty$, both sides are $\delta_{\infty}$. Otherwise, by the binomial theorem,

$$\text{Pois}(\sigma + \tau)(n) = e^{-(\sigma + \tau)} \left(\frac{\sigma + \tau}{n!}\right)^n = e^{-\sigma - \tau} \sum_{i+j=n} \frac{\sigma^i \tau^j}{i!j!} = \sum_{i+j=n} \text{Pois}(\sigma)(i)\text{Pois}(\tau)(j) = (\text{Pois}(\sigma) * \text{Pois}(\tau))(n).$$

Note that

$$\begin{array}{ccc}
\mathbb{N}^2 & \xrightarrow{\oplus} & \mathbb{N} \\
\mathcal{N}(2) & \xrightarrow{f^*} & \mathcal{N}(1)
\end{array}$$

where $f : 2 \to 1$.

Thus the above says: for any $\rho = (\sigma, \tau) \in \mathcal{M}(2) = \mathbb{N}^2$,

$$\text{Pois}(f^*(\rho)) = f^* \left(\text{Pois}(\sigma) \times \text{Pois}(\tau)\right).$$

An easy induction/approximation argument shows the obvious generalization to $\rho \in \mathbb{N}^n$, $n \leq \omega$. 

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For countable \( X \). Given \( \rho \in \mathcal{M}(X) \cong [0, \infty)^X \),
\[
\text{Pois}_X(\rho) := \prod_{x \in X} \text{Pois}(\rho(x)) \in \mathcal{M}_1(\mathbb{N}^X) \cong \mathcal{M}_1(\mathcal{N}(X)).
\]

The above Proposition, generalized to countable arity, now says that for \( f : X \to 1 \),
\[
\text{Pois}_1(f_*(\rho)) = f_{**}(\text{Pois}_X(\rho)).
\]

By taking a countable disjoint union, this also holds for \( f : X \to Y \) between two countable sets.

**Example.** Let \( G \) be a countable group, \( \rho \in \mathcal{M}(G) \) be counting measure.

Each \( g : G \to G \) preserves \( \rho \), i.e., \( g_*(\rho) = \rho \), whence
\[
\text{Pois}_G(\rho) = \text{Pois}_G(g_*(\rho)) = g_{**}(\text{Pois}_G(\rho)).
\]

So we have a pmp \( G \curvearrowright (\mathcal{N}(G), \text{Pois}_G(\rho)) = (\mathbb{N}^G, \text{Pois}(1)^G) \), where
\[
\text{Pois}(1) = (1/e, 1/e, 1/2e, 1/6e, \ldots) \quad (\text{supported on } \mathbb{N}).
\]

**For general standard Borel \( X \).** Let \( \rho \in \mathcal{M}(X) \). Consider all Borel “approximations” \( f : X \to Y \) by countable sets \( Y \). In other words, we are looking at countable Borel partitions \( X = \bigsqcup_{y \in Y} f^{-1}(y) \).

Note that this is an *uncountable* inverse system. It is nonetheless easily seen that
\[
X \cong \lim_{\leftarrow f : X \to Y} Y,
\]
\[
\mathcal{N}(X) \cong \lim_{\leftarrow f : X \to Y} \mathcal{N}(Y).
\]

The **Poisson process** \( \text{Pois}_X(\rho) \in \mathcal{M}_1(\mathcal{N}(X)) \) is the unique measure such that for each \( f : X \to Y \),
\[
\text{Pois}_Y(f_*(\rho)) = f_{**}(\text{Pois}_X(\rho)).
\]

In other words, if we pick a \( \text{Pois}_X(\rho) \)-random multiset \( \nu \in \mathcal{N}(X) \), then for any countable Borel partition \( f : X \to Y \), the collection of numbers \( \langle \nu(f^{-1}(y)) \rangle_{y \in Y} = f_*(\nu) \in \mathcal{N}(Y) = \mathbb{N}^Y \) will be distributed according to \( \text{Pois}_Y(f_*(\rho)) = \prod_{y \in Y} \text{Pois}(\rho(f^{-1}(y))) \).

**Proposition.** If \( \rho \) is \( \sigma \)-finite, then such a measure exists (and is automatically unique).

(Existence requires proof, because the above inverse system is uncountable; otherwise we could just apply the Kolmogorov consistency theorem.)
Remark. Even assuming $\rho$ is $\sigma$-finite, $\text{Pois}_X(\rho)$ by default lives on the non-standard Borel $\mathcal{N}(X)$. However, for each countable Borel partition $f : X \to Y$ such that each $\rho(f^{-1}(y)) < \infty$, we have

$$\text{Pois}_X(\rho)(\{\nu \in \mathcal{N}(X) \mid \forall y \,(\nu(f^{-1}(y)) < \infty)\}) = f_*(\text{Pois}_X(\rho))(\mathbb{N}^Y \subseteq \bar{\mathbb{N}}^Y = \mathcal{N}(Y))$$

thus $\text{Pois}_X(\rho)$ lives on the standard Borel subspace

$$\mathcal{N}_{(f^{-1}(y))_{y<\infty}}(X) := \{\nu \in \mathcal{N}(X) \mid \forall y \,(\nu(f^{-1}(y)) < \infty)\} \cong \prod_{y \in Y} \mathcal{N}_{<\infty}(f^{-1}(y)).$$

In most concrete situations, there will not be a canonical such partition. However, note that this space is the same as $\mathcal{N}_{\mathcal{I}^<\infty}(X)$, where $\mathcal{I}$ is the ideal generated by the partition. Often, there will be a canonical countably generated such ideal $\mathcal{I}$ with $X = \bigcup \mathcal{I}$ witnessing $\sigma$-finiteness of $\rho$.

Example. For locally compact Polish $X$, we can take $\mathcal{I}$ to be the ideal of precompact Borel sets; then any locally finite $\rho$ yields $\text{Pois}_X(\rho)$ living on the canonical standard Borel subspace $\mathcal{N}_{\mathcal{I}^<\infty}(X) \subseteq \mathcal{N}(X)$ of “locally finite multisets”.

Proposition. If $\rho$ is atomless, then $\text{Pois}_X(\rho)$ lives on

$$\mathcal{N}_{\text{set}}(X) := \{\nu \in \mathcal{N}(X) \mid \exists \text{ctbl } C \subseteq X \text{ s.t. } \nu = \sum_{c \in C} \delta_c\}.$$

Proof. Let $f : X \to Y$ be a countable Borel partition. Then $\text{Pois}_X(\rho)(\mathcal{N}_{\text{set}}(X))$ is at least

$$\text{Pois}_X(\rho)(\{\nu \in \mathcal{N}(X) \mid \forall y \,(\nu(f^{-1}(y)) \leq 1)\}) = f_*(\text{Pois}_X(\rho))(\{0, 1\}^Y \subseteq \bar{\mathbb{N}}^Y = \mathcal{N}(Y))$$

$$= \text{Pois}_Y(f_*(\rho))(\{0, 1\}^Y)$$

$$= \prod_{y \in Y} \text{Pois}(\rho(f^{-1}(y)))(\{0, 1\})$$

$$= \prod_{y \in Y} e^{-\rho(f^{-1}(y))}(1 + \rho(f^{-1}(y)))$$

$$= \exp\left(\sum_{y \in Y} (-\rho(f^{-1}(y)) + \log(1 + \rho(f^{-1}(y))))\right)$$

$$\geq \exp\left(-\sum_{y \in Y} \rho(f^{-1}(y))^2 / 2\right).$$

Since $\rho$ is atomless, we can choose $f$ so that $\sum_y \rho(f^{-1}(y))^2 \to 0$. □

Proposition. For any Borel automorphism $T : X \to X$ such that $T_* \rho = \rho$, we have

$$\text{Pois}_X(\rho)(\{\nu \in \mathcal{N}(X) \mid T_\# \nu \neq \nu\}) \geq 1 - \|\text{Pois}(\rho(\{x \in X \mid T x \neq x\})\|_2$$

where $r \mapsto 1 - \|\text{Pois}(r)\|_2 : [0, \infty] \to [0, 1]$ is an order-isomorphism.

In particular, if $T$’s set of non-fixed points had infinite $\rho$-measure, then $T_*$ is free $\text{Pois}_X(\rho)$-a.e.