One-ended spanning trees and generic combinatorics

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Based on joint work with Poulin and Zomback
Throughout $G$ will be a graph with bounded (finite) maximum degree $\Delta(G)$.

- A connected graph is **one-ended** if for every finite $F \subset V(G)$ the induced subgraph on $V(G) \setminus F$ has one infinite component.
Classical results

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- If $G$ has a one-ended spanning trees, there’s an easy proof that $\chi(G) \leq \Delta(G)$. 
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Can we find definable analogs of the above?
Borel graphs and Baire measure

Fix from now on a Polish space \((X, \tau)\).

- A graph \(G\) with \(V(G) = X\) is a **Borel graph** if \(E(G) \subset X^2\) is Borel.

- A subset of \(A \subseteq X\) is **nowhere dense** if \(\overline{A}\) has empty interior, it is **meagre** if it is a countable union of nowhere dense sets, and it is **Baire measurable** if there is an open set \(U\) and meagre set \(M\) with \(A = U \triangle M\).

- We say that \(G\) is one-ended if each of its connected components is.

- We say that \(G\) **admits a one-ended spanning tree generically** if there is a \(G\)-invariant comeagre Borel set \(X'\) and a Borel \(T \subset G|_{X'}\) such that \(T\) is acyclic, one-ended, and spans each \(G\) component that it meets.
Known results

- (Conley, Marks, Tucker-Drob ’16) Any Borel graph $G$ with no 0 or 2 ended components has a one-ended spanning subforest generically.

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  - This implies the Baire measurable Brook's theorem.

- (Timar '19. Conley, Gaboriau, Marks, Tucker-Drob '21) Every one-ended Borel graph has a one-ended spanning tree a.e.
  - (B., Kun, Sabok '21) This is useful for showing that $d$-regular bipartite graphs have Borel perfect matchings a.e., and that $2d$-regular graphs admit Borel balanced orientations.
New results

- (B., Poulin, Zomback '22+) Every bounded degree Borel graph admits a one-ended spanning tree generically.
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This is very useful for showing that $d$-regular bipartite graphs have Borel perfect matching generically, and $2d$-regular graphs have Borel balanced orientations generically.
Definition

A borel family of sets $\mathcal{T} \subset V(G)^{<\infty}$ is a toast if it satisfies properties (1) and (2) of the below definition, and it is a connected toast if it also satisfies property 3:

1. $\bigcup_{K \in \mathcal{T}} E(K) = E(G)$,
2. for every pair $K, L \in \mathcal{T}$ either $(N(K) \cup K) \cap L = \emptyset$ or $K \cup N(K) \subseteq L$, or $L \cup N(L) \subseteq K$,
3. for every $K \in \mathcal{T}$ the induced subgraph on $K \setminus \bigcup_{K \supseteq L \in \mathcal{T}} L$ is connected.
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- (Brandt, Chang, Grebik, Grunau, Rozhon, Vidnyanszky 21') Every bounded degree Borel graph admits a toast generically.

- (B., Kun, Sabok '21) Every one-ended Borel graph admits a connected toast a.e.
Theorem (B., Poulin, Zomback '22+)

*Every one-ended bounded degree Borel graph admits a connected toast generically.*

- Fix a toast $\mathcal{T}$ and order elements of $\mathcal{T}$ by inclusion, where $\mathcal{T}_1$ is the set of minimal elements, etc.
- For every $L \in \mathcal{T}_1$ there is an $m \in \omega$ and $L \subset K \in \mathcal{T}_{<m}$ such that $K \setminus L$ is connected. Here, we say $L$ is covered by level $m$. 
Theorem (B., Poulin, Zomback '22+)

Every one-ended bounded degree Borel graph admits a connected toast generically.

- Fix a toast $T$ and order elements of $T$ by inclusion, where $T_1$ is the set of minimal elements, etc.
- For every $L \in T_1$ there is an $m \in \omega$ and $L \subset K \in T_{<m}$ such that $K \setminus L$ is connected. Here, we say $L$ is covered by level $m$.
- For some $m \{x \in L \in T_1 : L$ is covered by level $m\}$ is non-meagre in $U_1$. Call the family of such $L$ tiles $A'$.
By the PHP we can choose one $A'_1$ tile from each maximal $T_{<m}$ tile so that the chosen tiles are still non-meagre in $U_1$. Call the chosen family of tiles $A_1$.
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the induced subgraph on $X \setminus V(A_1)$ is still one-ended.
Theorem

Any one-ended bipartite $d$-regular Borel graph admits a Borel perfect matching generically.
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- A fractional perfect matching is a function $\sigma : E(G) \to [0, 1]$ such that $\sum_{v \in e} \sigma(e) = 1$ for all $v \in V(G)$. 

\[ \text{Graph diagram} \]
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- **A fractional perfect matching** is a function $\sigma : E(G) \to [0, 1]$ such that $\sum_{e \ni v} \sigma(e) = 1$ for all $v \in V(G)$.

- $\sigma = \frac{1}{d}$ is a fractional perfect matching.

- Let $F(\sigma) = \sigma^{-1}(0, 1)$.
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- $F(\sigma)$ has no leaves.

- We can always fix one edge on a cycle.
Let $T$ be a connected toast. For every $L \in \mathcal{T}_1$ there is an $m \in \omega$, an $L \subset K \in \mathcal{T}_{<m}$, and a fractional matching $\sigma'$ such that

1. $\sigma'(e) \in \{0, 1\}$ for all $e \in E(L)$.
2. $\sigma'(e) = \sigma(e)$ for all $e \notin E(K)$.
Let $\mathcal{T}$ be a connected toast. For every $L \in \mathcal{T}_1$ there is an $m \in \omega$, an $L \subset K \in \mathcal{T}_{<m}$, and a fractional matching $\sigma'$ such that

1. $\sigma'(e) \in \{0, 1\}$ for all $e \in E(L)$.
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For every $e \in L$ and $L \subset K \in \mathcal{T}$ there's a cycle in $F(\sigma)$ that's a subset of $K$ and contains $e$. 
Problems

- Does every one-ended bipartite Borel graph satisfy $\chi_{BM}' \leq \Delta(G)$?

- Does every bipartite $d$-regular Borel graph that admits a connected toast have a Borel perfect matching?