Determinacy, measure, toasts, and the shift graph

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McGill DDC Seminar
Assume that $G$ is a graph and $\mathcal{V}(G)$ is endowed with a Borel structure. $n \in \{1, 2, \ldots, \aleph_0\}$ is equipped with the trivial Borel structure.
On Borel combinatorics

Assume that $G$ is a graph and $V(G)$ is endowed with a Borel structure. $n \in \{1, 2, \ldots, \aleph_0\}$ is equipped with the trivial Borel structure.

Can talk about:

*Borel graphs:* $G$ is a Borel graph is $G$ is Borel a subset of $V(G) \times V(G)$.

*Borel chromatic numbers:* minimal $n$ for which $G$ has a Borel $n$-coloring. Notation: $\chi_B(G)$.

*Borel homomorphisms:* if $G, H$ are Borel graphs, a Borel homomorphism is a Borel map $f : V(G) \to V(H)$ that takes edges to edges. Notation: $G \leq_B H$. 
The shift graph

Let us denote by $[S]^\infty$ the collection of countably infinite subsets of the set $S$. 
The shift graph

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**Theorem.** (Galvin-Prikry) Let \([\mathbb{N}]^\mathbb{N} = B_0 \cup \cdots \cup B_n\) be a Borel covering. Then there exists some \(i \leq n\) and \(A \subset \mathbb{N}\) infinite with \([A]^\mathbb{N} \subset B_i\).
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Let $S : [\mathbb{N}]^\mathbb{N} \rightarrow [\mathbb{N}]^\mathbb{N}$ be the *shift-map*, defined by

$$S(x) = x \setminus \{\min x\}.$$ 

Define the shift-graph $G_S$ by letting $x G_S y$ iff $y = S(x)$.
The shift graph

**Theorem.** (Kechris-Solecki-Todorčević) \( \chi_B(G_S) = \aleph_0 \).

If: IF \( c : \Sigma_0 \rightarrow \eta \) is BOREL
THEN \( (c^{-1}(i;i^2))_{i \leq n} \) is a BOREL covering

\( \theta \in \Sigma_0 \) AND \( Y \) G-P \( \exists A \in \Sigma_0 \)

\( \left[ A \right]^n \leq c^{-1}(i;i^2) \) \( \land \) \( C(A) = C(S(A)) = i \).

KST: \( \chi_B(G) > \aleph_0 \) \( \iff \) \( G \leq B \) \( G \)
The shift graph

**Theorem.** (Kechris-Solecki-Todorčević) \( \chi_B(G_S) = \aleph_0 \).

**Question.** Assume that \( G \) is an acyclic Borel graph with \( \chi_B(G) \geq \aleph_0 \). Is \( G_S \leq_B G \)?
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**Theorem.** (Pequignot) No.

**Theorem.** (Todorčević-V) There is no meaningful characterization of Borel graphs with Borel chromatic number $< n$, for each $n \in \{4, \ldots, \aleph_0\}$. 
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In fact, the set of such graphs is $\Sigma^1_2$-complete.
Gadgets and measures

**Theorem.** (Grebík-V.) There is some $d$ for which acyclic $d$-regular Borel graphs with Borel chromatic number $\leq 3$ form a $\Sigma^1_2$-complete set.
Gadgets and measures

2) **Cyclic** \(\Rightarrow\) **Acyclic**

\( (x, y) \in \mathcal{H} \times G^* \) \( (x', y') \in \mathcal{H} \times G^* \)

\( \Leftarrow \Rightarrow x \in \mathcal{H} \times G^* \) \( y \in \mathcal{B} \)

\( \mathcal{H} \times G^* \leq \mathcal{H} \leq \mathcal{B} \leq G^* \)

\( (\mathcal{H} \times G^*) \leq \mathcal{B} \leq \mathcal{B} \leq G^* \)

If \( \mathcal{H} \) is acyclic

\( \mathcal{H} \times \mathcal{B} \) contains only even cycles

\( \chi_B (G_B^*) \leq 3 \) \( \Leftarrow \Rightarrow \chi_B (G_B^* \times \mathcal{H}) \leq 3 \)

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Theorem. There exists a $d$, an acyclic $d$-regular Borel graph $\mathcal{H}$ on a probability measure space $(X, \mu)$ such that for every $B \subseteq X$ Borel with $\mu(B) \geq \frac{1}{3}$ we have $\mu(N_{\mathcal{H}}(B)) > \frac{2}{3}$. 
Gadgets and measures

**Theorem.** There exists a \( d \), an acyclic \( d \)-regular Borel graph \( \mathcal{H} \) on a probability measure space \((X, \mu)\) such that for every \( B \subseteq X \) Borel with \( \mu(B) \geq \frac{1}{3} \) we have \( \mu(N_\mathcal{H}(B)) > \frac{2}{3} \). In particular, if \( B, B' \subseteq X \) are measurable and with \( \mu(B), \mu(B') \geq \frac{1}{3} \) then there exist \( z \in B \) and \( z' \in B' \) that are adjacent in \( \mathcal{H} \).

(\text{MARKS}) \exists \ & \ & 3 \text{ REGULAR ACYCLIC BOREL GRAPH WITH } \chi_B(G) = 4
Theorem. (Todorčević-V) There is no meaningful characterization of Borel graphs with Borel chromatic number at most $n$, for each $n \in \{3, \ldots, \aleph_0\}$: such graphs form a $\Sigma^1_2$-complete set.
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(di Prisco-Todorčević-Miller) Assume that $B \subset [\mathbb{N}]^\mathbb{N}$ is a Borel set, and for some $\mathcal{G}_S$-independent Borel set $B'$, for each $x \in B$ there is $n$ with $S^n(x) \in B'$. Call such $B'$ and *independent hitting set.*
Complexity on the shift

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A set $S \subset [\mathbb{N}]^\mathbb{N}$ is called *non-dominating* if there is an $f \in [\mathbb{N}]^\mathbb{N}$ such that for each $g \in S$ we have $|\{n : g(n) \leq f(n)\}| = \aleph_0$. 

\[ f \begin{array}{c} \nearrow \hfill \searrow \hfill \swarrow \hfill \nwarrow \end{array} \begin{array}{c} g \in S \end{array} \]
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Non-dominating Borel sets admit an independent hitting set.
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Non-dominating Borel sets admit an independent hitting set.

**Theorem.** If a Borel coloring problem is solvable on non-dominating Borel sets, and not solvable on $[\mathbb{N}]^\mathbb{N}$, then the Borel subgraphs of $G_S$ on which it is solvable form a $\Sigma^1_2$-complete.
Shift and determinacy

**Theorem.** (B-C-G-G-R-V) Let $\mathcal{H}$ be a locally countable Borel graph. Then we have

$$\chi_{w\Delta_2}(\mathcal{H}) > 3 \Rightarrow \chi_B(\text{Hom}(T_3, \mathcal{H})) > 3.$$
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Thus, if $B \subset [\mathbb{N}]^\mathbb{N}$ is non-dominating then

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Thus,

- If $B \subset [\mathbb{N}]^\mathbb{N}$ is non-dominating then
  \[ \chi_B(\text{Hom}(T_3, \mathcal{G}_S \upharpoonright B)) \leq 3, \]
- \[ \chi_B(\text{Hom}(T_3, \mathcal{G}_S)) = 4. \]
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Thus,

- If $B \subset [\mathbb{N}]^\mathbb{N}$ is non-dominating then $\chi_B(\text{Hom}(T_3, G_S \upharpoonright B)) \leq 3$,
- $\chi_B(\text{Hom}(T_3, G_S)) = 4$.

**Theorem.** 3-regular acyclic Borel graphs with Borel chromatic number $\leq 3$ form a $\Sigma^1_2$-complete set.
Marks’ method
Toasts

Let $\mathcal{G}$ be a locally countable Borel graph and $k$. A $k$-toast is a sequence of Borel set $B_0 \subset B_1 \subset \cdots$ with

- $\bigcup_i B_i = V(G)$,
- $\mathcal{G} \upharpoonright B_i$ has finite connected components,
- if $S_i \neq S_j$ are connected components of some $\mathcal{G} \upharpoonright B_i$ and $\mathcal{G} \upharpoonright B_j$ then the distance of their boundaries is at least $k$. 

\[ \begin{array}{c}
B_0 \subset B_1 \subset \cdots
\end{array} \]
**Theorem.** Let $l$ be odd. Then a $k$-toastable acyclic Borel graph admits a Borel homomorphism into $C_l$ for every large enough $k$. 
Toasts and non-dominating sets

**Theorem.** Let $B \subset [\mathbb{N}]^\mathbb{N}$ be non-dominating. Then $G_S \upharpoonright B$ is $k$-toastable for each $k$. 
Toasts and non-dominating sets

**Theorem.** Let $B \subset [\mathbb{N}]^\mathbb{N}$ be non-dominating. Then $\mathcal{G}_S \upharpoonright B$ is $k$-toastable for each $k$.

**Theorem.** Toastable subgraphs of $\mathcal{G}_S$ form a $\Sigma^1_2$-complete set.
Theorem. (Hell-Nesetřil) Let $H$ be a finite graph. Deciding whether a finite graph $G$ admits a homomorphism into $H$ is $\Sigma^1_2$-complete, unless $H$ is bipartite.
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**Theorem.** (Thornton) Let $H$ be a finite graph. The Borel graphs that admit a Borel homomorphism to $H$ form a $\Sigma_2^1$-complete set, unless $H$ is bipartite, in which case this set is $\Pi^1_1$. 
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Theorem. Assume that $H$ contains an odd cycle. Then Borel subgraphs of $\mathcal{G}_S$ that admit a Borel homomorphism to $H$ form a $\Sigma^1_2$-complete set.
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Theorem. Assume that $H$ contains an odd cycle. Then Borel subgraphs of $G_\mathcal{S}$ that admit a Borel homomorphism to $H$ form a $\Sigma^1_2$-complete set.

Theorem. (C-M-S-V) There is a Borel graph $L$ with $\chi_B(G) > 2 \iff L \leq_B G$.
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**Theorem.** Assume that $H$ contains an odd cycle. Then Borel subgraphs of $G_S$ that admit a Borel homomorphism to $H$ form a $\Sigma^1_2$-complete set.

**Theorem.** (C-M-S-V) There is a Borel graph $L$ with $\chi_B(G) > 2 \iff L \leq_B G$. The Borel graphs that admit a Borel homomorphism to a bipartite graph form a $\Pi^1_1$ set.
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Combining the above theorems, we obtain a new, algebra-free strengthening of Thornton’s result.
Open questions

- Is the collection of compact free subshifts of $2^{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2}$ with Borel chromatic number $\leq 3$ also $\Sigma^1_2$-complete?
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- Is toastability $\Sigma^1_2$-complete on bounded degree acyclic Borel graphs?
- What are the Borel CSP’s that are solvable from toasts?
Thank you for your attention!