The group of absolutely continuous homeomorphisms of $[0,1]$ is topologically 2-generated

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Topological generating sets

Definition
Let $G$ be a Polish group, and let $1 \leq n \leq \aleph_0$.

$\Omega_n := \{ (g_i)_{i < n} \in G^n : \langle g_i : i < n \rangle$ is dense in $G \}$.

We say $G$ is topologically $n$-generated (resp. generically $n$-generated) if $\Omega_n$ is non-empty (resp. comeagre).

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The topological rank (resp. generic rank) of $G$, denoted by $\text{trk}(G)$ (resp. $\text{grk}(G)$), is the least $n$ for which $G$ is topologically $n$-generated (resp. generically $n$-generated).
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• Every Polish group is generically $\aleph_0$-generated. (By separability.)
• $\Omega_n$ is a $G_\delta$ set in $G^n$. Thus, $G$ is generically $n$-generated iff $\Omega_n$ is dense in $G^n$.
• If $\phi : G_1 \to G_2$ is a continuous group homomorphism with dense image, then $\text{trk}(G_2) \leq \text{trk}(G_1)$. 
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Example

Topologically 1-generated groups are also called monothetic. \((\mathbb{R}/\mathbb{Z})^n\) has this property for all \(n\), as does the group \(L_0(T)\).

These groups are also generically monothetic, i.e. generically 1-generated.

Example

\(\mathbb{R}^n\) is generically \((n+1)\)-generated.

Example (Kechris–Rosendal, 2007)

\(S_\infty\) is topologically 2-generated, as are many other automorphism groups of countable structures.

However, a non-archimedean group can never be generically \(n\)-generated for any finite \(n\).
Topological generating sets - examples

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$D^1_+(I)$ is topologically 10-generated. (The actual value of $\text{trk}(D^1_+)$ is likely lower, but it must be at least 3.)
Absolute continuity

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$$\sum_{i < n} b_i - a_i < \delta \implies \sum_{i < n} |f(b_i) - f(a_i)| < \epsilon.$$
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\sum_{i < n} (b_i - a_i) < \delta \quad \Rightarrow \quad \sum_{i < n} |f(b_i) - f(a_i)| < \epsilon.
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Every Lipschitz continuous function is absolutely continuous, and every absolutely continuous function has bounded variation.
Figure: The Cantor staircase is the canonical example of a non-abs cts function.
Theorem (Fundamental Theorem of Calculus for absolutely continuous functions)

For a function $f: I \to \mathbb{R}$, the following are equivalent:

(i) $f$ is absolutely continuous;

(ii) $f$ is differentiable almost everywhere, $f' \in L^1$, and we have $f(x) = f(0) + \int_0^x f'(t) \, dt$ for all $x \in I$;

(iii) There exists a map $g \in L^1$ such that $f(x) = f(0) + \int_0^x g(t) \, dt$ for all $x \in I$. 
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Absolutely continuous homeomorphisms

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The group $H_{AC}^+$ is the subgroup of $H^+$ given by:

$$H_{AC}^+ := \{ f \in H^+ : f \text{ and } f^{-1} \text{ are absolutely continuous} \}.$$ 

Equip $H_{AC}^+$ with the metric $d_{AC}(f, g) := \| f' - g' \|_1$. Thus, the map $H_{AC}^+ \ni f \mapsto f' \in L^1$ is an isometry.

Theorem (Solecki, 1995)
The metric $d_{AC}$ induces a Polish topology on $H_{AC}^+$, which is finer than the one inherited from $H^+$. 
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Aside: subgroups of $H_+$
Theorem (Akhmedov–Cohen, 2019)

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Suffices to show $\Omega_2$ is dense. Fix $f, g \in H_+$, and $\epsilon > 0$. We will build $\tilde{f}, \tilde{g} \in H_+$ such that $d\left(f, \tilde{f}\right) < \epsilon$, $d\left(g, \tilde{g}\right) < \epsilon$, and $\Gamma := \langle \tilde{f}, \tilde{g} \rangle$ is dense in $H_+$. 
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The set $\{(f, g) : \text{Fix}(f) \cap \text{Fix}(g) = \{0, 1\}\}$ is dense, so without loss of generality, we assume $f$ and $g$ do not share any fixed points in $(0, 1)$. 
Constructing \( \tilde{f} \) and \( \tilde{g} \)

Sketch of the construction of \( \tilde{f} \) and \( \tilde{g} \):

- Fix \( \alpha > 0 \) small
- Let \( \tilde{g} \) have the following properties:
  - \( \tilde{g} \) agrees with \( g \) on \( [\alpha, 1] \)
  - There is \( y_0 \in (0, \alpha) \) such that \( \tilde{g}(y_0) = y_0 \) and \( \tilde{g}(x) > x \) for all \( x \in (0, y_0) \).
- Fix an arbitrary \( x_0 \in (0, y_0) \), and let \( x_n = \tilde{g}^{-n}(x_0) \).
- Fix elements \( \phi_0, \phi_1 \in H^+(x_1, x_0) \) that generate a dense subgroup of \( H^+(x_1, x_0) \).
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  - \( \tilde{f}(x) > x \) for all \( x \in (x_0, y_0] \)
  - On \( [x_n+1, x_n] \), \( \tilde{f} \) agrees with \( \tilde{g}^{-n} \circ \phi_0 \circ \tilde{g}^n \) for \( n \) even and \( \tilde{g}^{-n} \circ \phi_1 \circ \tilde{g}^n \) for \( n \) odd.
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Why this works

• For any \( n \), \( \tilde{f} \downarrow [x^n+1, x^n] \)
  and \( \tilde{g} \downarrow [x^n+1, x^n] \) generate a dense subgroup of \( H^+(\mathbb{R}) \).

• \( \tilde{f} \) and \( \tilde{g} \) do not share any fixed points. Thus, for any \( x > 0 \) and \( y < 1 \), there is \( h \in \Gamma \) such that \( h(x) > y \).

• Using this, one shows that for any \( \lambda > 0 \), there is \( \Phi \in \Gamma \) and some \( [a, b] \subseteq \mathbb{R} \) such that
  \( a < \lambda < 1 - \lambda < b \), and \( \Phi \tilde{f} \Phi^{-1} \downarrow [a, b] \) and \( \Phi \tilde{g} \Phi^{-1} \downarrow [a, b] \) generate a dense subgroup of \( H^+(\mathbb{R}) \).
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- Using this, one shows that for any $\lambda > 0$, there is $\Phi \in \Gamma$ and some $[a, b] \subseteq I$ such that $a < \lambda < 1 - \lambda < b$, and $\Phi\tilde{f}\Phi^{-1}|_{[a, b]}$ and $\Phi\tilde{g}\tilde{f}\tilde{g}^{-1}\Phi^{-1}$ generate a dense subgroup of $H_+ ([a, b])$. 
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- Need to show $F := \{(f, g) : \text{Fix}(f) \cap \text{Fix}(g) = \{0, 1\}\}$ dense in $(H^{AC}_+)^2$.
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- Need to choose \( \alpha \) even smaller to guarantee \( d_{\text{AC}}(g, \tilde{g}) < \epsilon \).
- Need more \( \phi_i \)'s to generate a dense subgroup of \( H^\text{AC}_+([x_1, x_0]) \).
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- Need to choose $\alpha$ even smaller to guarantee $d_{AC}(g, \tilde{g}) < \epsilon$.
- Need more $\phi_i$’s to generate a dense subgroup of $H_{+}^{AC}([x_1, x_0])$.
- Need to show why a dense subgroup of $H_{+}^{AC}([a, b])$ can approximate $H_{+}^{AC}(I)$. 