One-ended spanning subforests
in pmp graphs of superquadratic growth

Last time...

- The number of ends of $G$ is the supremum of the number of infinite connected components in $G|(X \setminus F)$ for $F \subseteq G$ finite.

**Theorem**

If $G$ is hyperfinite, pmp, and almost nowhere 0 or 2 ended then $G$ has an a.e. one-ended treeing.

This time...

**Theorem (2.6)** Suppose $G$ is a pmp locally finite Borel graph on $(X, \mu)$ of superquadratic growth (i.e., $\exists c > 0$ s.t. $\forall r \exists r': 13r \leq c r'^2$) then $G$ has a Borel a.e. one-ended spanning subforest

Note: Isoperimetric constant $\varphi$ of $G$ is

$$\varphi := \inf_{A \subseteq X \setminus \{x\}} \frac{\mu(2 \cdot A)}{\mu(A)}$$

where $x \in X$, $\mu(A) > 0$, and $G|_A$ is compact.
If $\gamma > 0$ then $G_j$ has exponential growth, so Theorem applies.

**Lemma (2.5)** $G_j$ a loc finite pmp Borel graph on $(X, \mu)$ & there are partial Borel functions $f_0, f_1, \ldots \subseteq G_j$ s.t.

1. $\text{Udom}(f_i) = X$
2. $\sum \mu(\text{dom}(f_i)) < \infty$
3. Each $f_i$ is aperiodic & has finite back orbits, i.e., $\forall x \exists y \in X : \exists n \in \mathbb{N} : f_n^i y = x$ finite
4. $\forall i \& \forall x \in \text{dom}(f_i)$ there is $j \geq i$ with $f_i(x) \in \text{dom}(f_j)$

Then $G_j$ has a Borel a.e. one-ended spanning subforest

**Idea**: Put the $f_i$'s together to get a Borel (full) $f : X \to X \subseteq G_j$ aperiodic & has finite back orbits
Proof. \( B = \{ x : \exists i \in \mathbb{N} \text{ s.t. } x \in \text{dom}(f_i) \} \) is null

\[ = \limsup (\text{dom}(f_i)) \]

So we may assume WLOG that \( \forall x \in X \)
\[ \exists i \in \mathbb{N} : x \in \text{dom}(f_i) \] is nonempty & finite

Define \( f : X \to X \subseteq G \)

\[ x \mapsto f_{n(x)}(x) \]

where \( n(x) := \max \{ i \in \mathbb{N} : x \in \text{dom}(f_i) \} \)

By (4), \( n(x) \) is non-decreasing along \( f \)-orbits

\[ \begin{array}{c}
    x \\
    \xrightarrow{f} f_4 \\
    \xrightarrow{f} f_{f_4} \\
    \xrightarrow{f} f_{f_{f_4}} \\
    \xrightarrow{f} \vdots
\end{array} \]

So \( f \) is aperiodic by this fact & (3)

By (3), each \( f_i \) has finite backward orbits

so \( f \) does as well.
Theorem (2.6) Suppose $G$ is a pmp locally finite Borel graph on $(X, \mu)$ of super-quadratic growth (i.e., $\exists c > 0$ s.t. $\forall x \forall r \exists B_r(x) | \geq cr^2$) then $G$ has a Borel a.e. one-ended spanning subforest.

Proof: We'll come up with $f_i \leq G$ satisfying (1) - (4) from lemma.

Let $r_n := 2^n$

(s0 $\sum_{n=0}^{\infty} \frac{2^{r_{n+1}}}{c r_n^2} < \infty$)

Put $A_0 := X$

for $n > 1$ let $A_n$ be Borel & maximal wrt the fact that for $x \neq y \in A_n$,

$B_{r_n}(x) \cap B_{r_n}(y) = \emptyset$
Def. of \( f_n \): for \( x \in A_n \) take \( \text{lex} \)
least minimal length path from \( x \) to a point \( y \in A_{n+1} \)
\[ x = x_0, x_1, x_2, \ldots, x_k = y \]
Define \( f_n \) to be the union of the pairs \( (x_i, x_{i+1}) \) in these paths.

So \( f_i \leq G_j \) are partial Borel functions.

(1) holds since \( A_n \leq \text{dom}(f_n) \)
\[ A_0 = X \]

(3): aperiodic & finite back orbits
minimality of paths & lengths of paths are bounded (by maximality of \( A_n \)).

(4): \( x \in \text{dom}(f_i) \Rightarrow f_i(x) \in \text{dom}(f_j) \)
\[ j \geq i \]
If \( f_i(x) \) was in the middle of a path then \( f_i(x) \in \text{dom}(f_i) \)
Otherwise \( f_i(x) \in A_{i+1} \leq \text{dom}(f_{i+1}) \)
(2) \[ \sum \mu(\text{dom}(f_n)) < \infty \]

By maximality of \( A_{n+1} \), the length of a path from \( x \in X \) to \( y \in A_{n+1} \)
\[ \leq 2r_{n+1} \]

\[ \Rightarrow \mu(\text{dom}(f_n)) \leq 2r_{n+1} \mu(A_n) \leq \frac{2r_{n+1}}{Cr_n} \]

since \( \mu(A_n) \leq \frac{1}{Cr_n} \)

Define \( E_n := \) being in the same ball of radius \( r_n \) for \( x \in A_n \)

\[ \int f(x) \, d\mu(x) = \int \text{Average of} \ f \ \text{over equivalence class of} \ x \]
Isoperimetric constant $\gamma > 0$

then we have exponential growth

$$\forall x \in X \quad |B_n(x)| \geq (1 + \gamma)^n$$

**Proof** by induction $|B_0(x)| = 1$

Suffices to show $|B_{n+1}(x)| \geq (1 + \gamma)|B_n(x)|$

for $\mu$-a.e. $x \in X$

If not, we can take a maximal disjoint collection of balls of radius $n+1$

(which $\leftrightarrow$ Borel) around points $x$

with $|B_{n+1}(x)| < (1 + \gamma)|B_n(x)|$

$$A := \bigcup_{x \in X} B_n(x) \quad \Rightarrow \quad A \text{ is Borel, positively measured}$$

$$\mu(\partial \delta A) \leq \gamma \mu(A) \implies \frac{\mu(\partial \delta A)}{\mu(A)} \leq \gamma$$

□