“One-ended spanning subforests and treeability of groups”
by Conley–Gaboriau–Marks–Tucker-Drob

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Preliminaries

- countable Borel equivalence relation (CBER) \( E \) (or \( R \)) \( \subseteq X^2 \): 
  \[ E \subseteq X^2 \text{ Borel, } (x)E \text{ is } 1 \text{-} \text{bl} \quad \forall x \in X. \]

- for Borel \( \Gamma \rhd X \), orbit equivalence relation \( E_\Gamma \) (or \( R_\Gamma \))
  \[ x E_\Gamma y : \iff \exists \gamma \in \Gamma \text{ s.t. } x = \gamma y \]

- for a Borel graph \( G \subseteq X^2 \), \( E_G \) (or \( R_G \))
  \[ x E_G y : \iff \exists x = x_0 x_1 \cdots x_n = y \]

- \( G \) is a graphing
- \( G \) is a treeing if furthermore, \( G \) is acyclic

Example: If \( \Gamma \rhd X \) and \( \Gamma = \langle S \rangle \)
  \[ x \rho y : \iff \exists i \in S \text{ s.t. } x = y \]

- for a Borel measure \( \mu \) on \( X \), a property of \( E \) holds \( \mu \)-a.e. if it holds on an \( E \)-in \( \mu \)-almost-null set.
Planar graphs are measure treeable

Theorem (CGMT 2021)

Let $G \subseteq X$ be a locally finite Borel planar* graph. Then for any Borel probability measure $\mu$ on $X$, $G$ has a Borel subtreeing $\mu$-a.e. In particular, $E_G$ is treeable $\mu$-a.e.

"each component is planar"

$\exists$ Borel acyclic $T \subseteq G$ s.t. $E_T = E_G$. 
A graph $G$ on vertices $X$ is **planar** if $\exists$ a planar embedding $f: X \to \mathbb{R}^2$, \&

$$
\begin{align*}
& f_e: [0,1] \to \mathbb{R}^2, \quad \text{for each } e = (x,y) \in E, \text{ which map distinct images except at endpoints.} \\
& \text{A facial cycle } (e_1, \ldots, e_n) \text{ is a cycle whose image under } f \text{ is a cut for a bounded } K \text{ of } \mathbb{R}^2 \text{ in } f.
\end{align*}
$$

- each edge belongs to at least 2 facial cycles
- the characteristic functions of facial cycles are linearly independent over $\mathbb{Z}/2\mathbb{Z}$
- (if $X$ is finite) they span the characteristic functions of all cycles over $\mathbb{Z}/2\mathbb{Z}$

A 2-basis in $G$ is a family of cycles obeying these 3 conditions.

**Theorem (Mac Lane 1937)**

If $X$ is finite, each 2-basis is the set of facial cycles for see planar embedding.
An **accumulation point** of a planar embedding $f : (X, G) \to \mathbb{R}^2$ is a point which is a limit of an infinite sequence of distinct vertices or edges.

**Example**

![Diagram of a planar graph with accumulation points indicated]

**Theorem (Thomassen 1980)**

- If a planar embedding has no accumulation points, the facial cycles form a 2-basis.
- If $G$ is locally finite, each 2-basis is given by a planar embedding.

A Borel graph $G \subseteq X^2$ is **Borel planar** if it has a Borel 2-basis as a subset of $[x]^{\mathbb{N}}$. 
Planar groups are measure treeable

Theorem (CGMT 2021)

Let $G \subseteq X$ be a locally finite Borel planar graph. Then for any Borel probability measure $\mu$ on $X$, $G$ has a Borel subtreeing $\mu$-a.e. In particular, $E_G$ is treeable $\mu$-a.e.

Corollary (CGMT 2021)

If $\Gamma$ acts freely on a connected planar graph with equivariant 2-basis, then every free Borel $\Gamma$-action is treeable $\mu$-a.e. for every Borel probability measure $\mu$. 

\[ \Gamma \text{ is measure-strongly-treeable} \implies \Gamma \text{ strongly-treeable} \implies \Gamma \text{ treeable} \]

Every proper action is treeable 

\[ \text{if a proper treeable action} \]
Let $\Sigma$ be a closed orientable surface. Its fundamental group $\pi_1(\Sigma)$ is

\[ \langle a_1, b_1, \ldots, a_n, b_n \rangle / \text{its relations} \]

\[ a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots \]

Every surface $\Sigma$ (except $S^2$) is a free quotient $\tilde{\Sigma}/\pi_1(\Sigma)$ where $\tilde{\Sigma} \cong \mathbb{R}^2$.

**Corollary (CGMT 2021)**

Every free Borel action of $\pi_1(\Sigma)$ is treeable $\mu$-a.e.
Theorem (CGMT 2021)

Let $G \subseteq X$ be a locally finite Borel planar graph. Then for any Borel probability measure $\mu$ on $X$, $G$ has a Borel subtreeing $\mu$-a.e. In particular, $E_G$ is treeable $\mu$-a.e.

A group $\Gamma$ is **elementarily free** if it is elementarily equivalent to $\mathbb{F}_2$.

Corollary (CGMT 2021)

Every free Borel action of a f.g. elementarily free group is treeable $\mu$-a.e.

Proof uses an explicit construction of a space $X$ with $\pi_1(X) \cong \Gamma$ (Sela 2006, Guirardel–Levitt–Sklinos 2020).
Some other treeable groups

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**Corollary (CGMT 2021)**

Every free Borel action of $\text{Isom}(\mathbb{H}^2)$ is treeable $\mu$-a.e.
Let $G$ be a graph on $X$. An **end** in $G$ is "

For each finite $F \subseteq X$, look at $\pi_0(G| (X \setminus F)) :=$

For finite $F_0 \subseteq F_1 \subseteq X$,

The **space of ends** of $(X, G)$ is $\partial G :=$

If $G$ is locally finite:
One-ended spanning subforests

Let $G \subseteq X^2$ be a Borel graph. A one-ended spanning subforest is

Conjecture (CGMT 2021)

Let $G \subseteq X^2$ be a locally finite Borel graph with $E_G \mu$-a.e. nonsmooth. TFAE:

(i) $G$ is $\mu$-a.e. not 2-ended.
(ii) $G$ has a Borel one-ended spanning subforest $\mu$-a.e.

►
► (CMT 2016)
► (CGMT 2021)
Cutting cycles along a one-ended subforest

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Proof idea:
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<td>For a compact surface $\Sigma$, every free Borel action of $\pi_1(\Sigma)$ is treeable $\mu$-a.e.</td>
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<td>For a compact aspherical $n$-manifold $M$, every free Borel action of $\pi_1(M)$ admits a “Borel family of contractible $(n - 1)$-dim’l simplicial complexes on each class”, up to $\mu$-a.e. Borel reducibility.</td>
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