Substitution on infinite alphabets and generalized Bratteli-Vershik models

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Plan for this talk

- Introduction to Bratteli diagrams and generalized Bratteli diagrams

- Two versions of Kakutani-Rokhlin tower construction for a class of substitution dynamical systems on a countably infinite alphabets (known as left determined substitution).

- Bratteli-Vershik (B-V) models for such substitution dynamical systems.

- Using the Bratteli-Vershik model we find explicit expressions for invariant and ergodic measures for such substitution dynamical system.

- Example. ✅
Let $\mathcal{A}$ be a countably infinite set, called an *alphabet*. We denote by $\mathcal{A}^\mathbb{Z}$ the bi-infinite sequence $(x_i)_{i \in \mathbb{Z}}$ on $\mathcal{A}$.

Note that $\mathcal{A}^\mathbb{Z}$ is a non-compact Polish space.

For $x \in \mathcal{A}^\mathbb{Z}$ we denote by $\mathcal{L}_n(x)$, the set of all words of length $n$ in $x$.

*Language* of $x$ is defined by $\mathcal{L}(x) := \bigcup_{n \in \mathbb{N}} \mathcal{L}_n(x)$.

A *substitution* $\sigma$ on $\mathcal{A}$ is a map from $\mathcal{A}$ to $\mathcal{A}^+$ (the set of finite non-empty words on $\mathcal{A}$), which associates to the letter $a \in \mathcal{A}$ the word $\sigma(a) \in \mathcal{A}^+$, with length $h_a := |\sigma(a)| < \infty$.

We define *language of a substitution* $\sigma$ by:

$\mathcal{L}_\sigma = \{\text{factors of } \sigma^n(a) : \text{ for some } n \geq 0, a \in \mathcal{A}\}$.

Define $X_\sigma = \{x \in \mathcal{A}^\mathbb{Z} : \mathcal{L}(x) \subset \mathcal{L}_\sigma\} \subset \mathcal{A}^\mathbb{Z}$.
The left shift $T : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$, is defined by $(Tx)_k = x_{k+1}$, for all $k \in \mathbb{Z}$.

$X_\sigma$ is a Polish space and is closed under $T$. We call $(X_\sigma, T)$ the subshift associate with substitution $\sigma$.

$(X_\sigma, T)$ is Borel dynamical system i.e. $T$ is a homeomorphism of Polish space.

For a finite string $\bar{x} = (x_0, ..., x_n)$ of length $n$, denote by $[\bar{x}]$ a cylinder set: $[\bar{x}] := \{y = (y_i) \in X_\sigma : y_0 = x_0, ..., y_n = x_n\}$.

We say that $\sigma$ is of bounded length if there exists an integer $C \geq 2$ such that for every $a \in \mathcal{A}$, $|\sigma(a)| \leq C$.

By identifying $\mathcal{A}$ with $\mathbb{Z}$, we say that $\sigma$ is of bounded size, if it is of bounded length and there exists a positive integer $t$ (independent of $n$ and minimal possible), such that for every $n \in \mathbb{Z}$, if $m \in \sigma(n)$, then $m \in \{n - t, ..., n, ..., n + t\}$.
**Definition** ([Ferenczi 2006]). We say that a substitution $\sigma$ on a countable alphabet $A$ is **left determined** if there exists $N \in \mathbb{N}$ such that, any word $w \in \mathcal{L}_\sigma$ of length at least $N$, has a unique decomposition $w = w_1 \ldots w_s$, such that each $w_i = \sigma(a_i)$ for unique $a_i \in A$, except that $w_1$ may be a suffix of $\sigma(a_1)$ and $w_s$ may be a prefix of $\sigma(a_s)$.

Example: The squared drunken man substitution:

$n \mapsto (n - 2) n n (n + 2); n \in 2\mathbb{Z}$

is left determined (see [F06]).
Definition ([Ferenczi 2006]). We say that a substitution $\sigma$ on a countable alphabet $A$ is left determined if there exists $N \in \mathbb{N}$ such that, any word $w \in L_\sigma$ of length at least $N$, has a unique decomposition $w = w_1 \ldots w_s$, such that each $w_i = \sigma(a_i)$ for unique $a_i \in A$, except that $w_1$ may be a suffix of $\sigma(a_1)$ and $w_s$ may be a prefix of $\sigma(a_s)$.

Example: The squared drunken man substitution:

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is left determined (see [F06]).
Example: a (non-simple, finite rank) Bratteli diagram

\( \pi, \rho : E \rightarrow V \)

\( \forall v \in V \setminus V_0 \quad \pi^{-1}(v) \neq \emptyset \)

\( \forall v \in V \quad \rho^{-1}(v) \neq \emptyset \)

\[ B(V,E) \]

\[ |V_i| < \infty \]

\[ |E_i| < \infty \]

\[ V = \bigcup_i V_i \]

\[ E = \bigcup_i E_i \]

\( (E_i)_i \)

\( \rho(e_i) \)

\( = \rho(e_{i+1}) \quad \forall i \)
A Bratteli diagram is a graded infinite graph \( B = (V, E) \) with the vertex set \( V = \bigsqcup_{i \geq 0} V_i \) and edge set \( E = \bigsqcup_{i \geq 1} E_i \):

1) \( V_0 = \{v_0\} \) is a single point;

2) \( V_i \) and \( E_i \) are finite sets for every \( i \);

3) edges \( E_i \) connect \( V_{i-1} \) to \( V_i \): there exist maps \( r \) (range) and \( s \) (source) from \( E \) to \( V \) such that \( r(E_i) \subseteq V_i \), \( s(E_i) \subseteq V_{i-1} \), and \( s^{-1}(v) \neq \emptyset \); \( r^{-1}(v') \neq \emptyset \) for all \( v \in V \) and \( v' \in V \setminus V_0 \).

- \( B \) is **stationary** if it repeats itself below the first level.
- \( B \) is of **finite rank** if for all \( n \geq 1 \), \( |V_n| \leq k \) for some positive integer \( k \).
- We say a finite rank diagram \( B \) has **rank** \( d \) if \( d \) is the smallest integer such that \( |V_n| = d \) infinitely often.
The incidence matrix $F_n$ is a $|V_n| \times |V_{n-1}|$ matrix with entries

$$f_{v,w}^{(n)} = |\{ e \in E_n : s(e) = w, r(e) = v \}|, \ v \in V_n, w \in V_{n-1}.$$  

A Bratteli diagram is called **simple** if $\forall n \exists m > n$ such that $F_m \cdots F_{n+1} > 0$ (all entries are positive).

A finite or infinite sequence of edges $(e_i : e_i \in E_i)$ such that $r(e_i) = s(e_{i+1})$ is called a **finite or infinite path**. Let $X_B$ be the set of infinite paths starting at the top vertex $v_0$. Then $X_B$ a 0-dimensional compact metric space w.r.t. the topology generated by cylinder sets

$$[\bar{e}] := \{ x \in X_B : x_i = e_i, \ i = 0, \ldots, n \}.$$
Incidence matrix (Example)

The diagram is *stationary* with incidence matrix

\[
F = \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 2 & 2 \\
\end{pmatrix}
\]

The sequence \( (F_n) \) of incidence matrices determine the structure of a Bratteli diagram.

**Topology on the path space** \( X_B \): two paths are close if they agree on a large initial segment.
Ordered Bratteli diagrams
Take a vertex $v \in V \setminus V_0$. 
Take a vertex $v \in V \setminus V_0$. Consider the set $r^{-1}(v)$. 
Ordered Bratteli diagrams

- Take a vertex $\nu \in V \setminus V_0$.
- Consider the set $r^{-1}(\nu)$.
- Enumerate edges from $r^{-1}(\nu)$.
Ordered Bratteli diagrams

- Take a vertex $v \in V \setminus V_0$.
- Consider the set $r^{-1}(v)$.
- Enumerate edges from $r^{-1}(v)$.
- Do the same for every vertex.
An infinite path $x = (x_n)$ is called **maximal** if $x_n$ is maximal in $r^{-1}(r(x_n))$. Similarly, **minimal** paths are defined.
An infinite path $x = (x_n)$ is called \textbf{maximal} if $x_n$ is \textit{maximal} in $r^{-1}(r(x_n))$. Similarly, \textbf{minimal} paths are defined.

The sets $X_{\text{max}}$ and $X_{\text{min}}$ of all maximal and minimal paths are non-empty and closed.
Define the Vershik map

\[ \varphi_B : X_B \backslash X_{\text{max}} \to X_B \backslash X_{\text{min}} : \]
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$$\varphi_B : X_B \setminus X_{\text{max}} \to X_B \setminus X_{\text{min}} :$$

Fix $$x \in X_B \setminus X_{\text{max}}.$$
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Fix \( x \in X_B \setminus X_{\text{max}} \).

Find the first \( k \) with non-maximal \( x_k \).
Define the Vershik map

\[ \varphi_B : X_B \setminus X_{\text{max}} \rightarrow X_B \setminus X_{\text{min}} : \]

Fix \( x \in X_B \setminus X_{\text{max}} \).

Find the first \( k \) with non-maximal \( x_k \).

Take \( x_k \) to its successor \( \bar{x}_k \).
Define the Vershik map

\( \varphi_B : X_B \setminus X_{\text{max}} \to X_B \setminus X_{\text{min}} : \)

Fix \( x \in X_B \setminus X_{\text{max}}. \)

Find the first \( k \) with non-maximal \( x_k. \)

Take \( x_k \) to its successor \( \overline{x}_k. \)

Connect \( s(\overline{x}_k) \) to the top vertex \( V_0 \)
by the minimal path.

\[ \varphi_B (\text{red}) = \text{blue} \]
Ordered Bratteli diagrams
Vershik map

- $\varphi_B$ is defined everywhere on $X_B \setminus X_{\text{max}}$
- $\varphi_B(X_B \setminus X_{\text{max}}) = X_B \setminus X_{\text{min}}$

Definition

If the map $\varphi_B$ can be extended to a homeomorphism of $X_B$ such that $\varphi_B(X_{\text{max}}) = X_{\text{min}}$, then $(X_B, \varphi_B)$ is called a Bratteli-Vershik system and $\varphi_B$ is called the Vershik map.

Question:
Under what conditions on a Bratteli diagram does the Vershik map exist?
Ordered Bratteli diagrams

Vershik map

- \( \varphi_B \) is defined everywhere on \( X_B \setminus X_{\text{max}} \)
- \( \varphi_B(X_B \setminus X_{\text{max}}) = X_B \setminus X_{\text{min}} \)

**Definition**

If the map \( \varphi_B \) can be extended to a homeomorphism of \( X_B \) such that \( \varphi_B(X_{\text{max}}) = X_{\text{min}} \), then \( (X_B, \varphi_B) \) is called a Bratteli-Vershik system and \( \varphi_B \) is called the Vershik map.

**Question:**

Under what conditions on a Bratteli diagram does the Vershik map exist?

**Answer:**

If a Bratteli diagram is simple, then the Vershik map always exists (e.g., use the left-to-right order).
Herman, Putnam, and Skau [HPS’92] showed that for every minimal Cantor dynamical system \((X, T)\), there exists a simple, ordered Bratteli diagram \(B\) such that the corresponding Vershik map \(\varphi_B\) is conjugate to \(T\).

Bezuglyi, Dooley and Medynets (2005), Medynets (2006) extended above result to \textit{aperiodic} Cantor dynamical systems.


Forrest (1997), Durand, Host, Skau (1999) described completely the class of dynamical systems that are represented by simple stationary Bratteli diagram. These are \textit{minimal substitution dynamical systems}.
Generalized Bratteli diagrams example

\[ \mathcal{U} = \bigcup_{i} V_i \]

\[ |x^{-1}(v)| < \infty \]

\[ Y_B = \bigcap_{i} E_i \]

\[ \mathcal{B} = \bigcup_{i} E_i \]
Generalized Bratteli diagrams (GBD)

Generalized Bratteli diagram $B = (V, E)$ is a countable graded graph $B = (V, E)$ with $V = \bigsqcup_{i \geq 0} V_i$ and $E = \bigsqcup_{i \geq 0} E_i$ such that,

(i) The set $V_i$, for $i \geq 0$ is countably infinite (identified with $\mathbb{Z}$). $E_i$ is the set of edges between the levels $V_i$ and $V_{i+1}$;

(ii) range map $r$ and source map $s$ from $E$ to $V$ such that $r(E_i) \subset V_i$, $s(E_i) \subset V_{i-1}$, $s^{-1}(v) \neq \emptyset$ for all $v \in V$, and $r^{-1}(v) \neq \emptyset$ for all $v \setminus V_0$;

(iii) for every $v \in V \setminus V_0$, the set $r^{-1}(v)$ is finite. For every $w \in V_i$, $v \in V_{i+1}$, the set of edges (denoted $E(w, v)$) between $w$ and $v$ is finite (or empty);

(iv) Put $|E(w, v)| = f_{vw}^i$. This defines a sequence of infinite incidence matrices $(F_n; n \in \mathbb{N}_0)$ whose entries are non-negative integers:

$$F_i = (f_{vw}^{(i)} : v \in V_{i+1}, w \in V_i), \quad f_{vw}^{(i)} \in \mathbb{N}_0.$$
Generalized Bratteli diagram (GBD) cont.

- If $F_n = F$ for all $n \in \mathbb{N}_0$, then diagram is called stationary.
- We denote by $Y_B$ the set of infinite paths in $B = (V, E)$.
- $Y_B$ is a Polish space using topology generated by cylinder sets $[e] := \{ x \in Y_B : x_i = e_i, \ i = 0, \ldots, n \}$.
- $B = (V, E)$ together with a linear order on $r^{-1}(v)$ for every $v \in V \setminus V_0$, is called an ordered generalized Bratteli diagram denoted by $B = (V, E, \geq)$.
- For an ordered diagram $B = (V, E, \geq)$, we define Vershik map $\varphi : Y_B \to Y_B$ as before.
- $(Y_B, \varphi)$ is a Borel dynamical system.

**Theorem (Bezuglyi, Dooley, Kwiatkowski (2006))**

Let $T$ be an aperiodic Borel automorphism of $(X, \mathcal{B})$. Then there exists an ordered generalized Bratteli diagram $B = (V, E, \geq)$ and a Vershik map $\varphi_B : Y_B \to Y_B$ such that $(X, T)$ is isomorphic to $(Y_B, \varphi_B)$. 
For \( a_i \in A; \ i = \{1, ..., n\} \), we denote by \([a_1...a_n]\) a cylinder set of length \( n \).

**Theorem 1.** Let \( \sigma \) be a *left determined substitution* on countably infinite alphabet \( A \) and \((X_\sigma, T)\) be the corresponding subshift. Then for any \( n \in \mathbb{N} \), we have a partition of \( X_\sigma \) into K-R towers

\[
X_\sigma = \bigsqcup_{a_1....a_n \in \mathcal{L}_n(\sigma)} \bigcup_{k=0}^{h_1+...+h_n-1} T^k[\sigma(a_1....a_n)]
\]

where \( h_k = |\sigma a_k| \) for \( k \in \{1, ..., n\} \).
For $a_i \in A; \ i = \{1, \ldots, n\}$, we denote by $[a_1 \ldots a_n]$ a cylinder set of length $n$.

**Theorem 1.** Let $\sigma$ be a *left determined substitution* on countably infinite alphabet $A$ and $(X_\sigma, T)$ be the corresponding subshift. Then for any $n \in \mathbb{N}$, we have a partition of $X_\sigma$ into K-R towers

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where $h_k = |\sigma a_k|$ for $k \in \{1, \ldots, n\}$.

**Theorem 2.** Let $\sigma$ be *left determined* and *bounded length* substitution on a countable infinite alphabet $A$, and $(X_\sigma, T)$ be the corresponding subshift. Then for every $n \in \mathbb{N}$, we have a partition of $X_\sigma$ into K-R towers

$$X_\sigma = \bigsqcup_{a_i \in A} \bigsqcup_{k=0}^{h_i^n - 1} T^k[\sigma^n a_i], \text{ where } |\sigma a_i| = h_i.$$
We used Theorem 1 and Theorem 2 to obtain:

**Corollary 3.** Let $\sigma$ be a left determined substitution of bounded size on countably infinite alphabet $\mathcal{A}$ and $(X_\sigma, T)$ be the corresponding subshift. Then there exists two sequence $(A_n)$ and $(B_n)$ of Borel sets with $A_0 = B_0 = X_\sigma$ and for $n > 0$,

$$A_n = \bigsqcup_{a_i \in \mathcal{A}} [\sigma^n a_i], \text{ and } B_n = \bigsqcup_{a_1 \ldots a_n \in \mathcal{L}_n(\sigma)} [\sigma(a_1 \ldots a_n)].$$

such that

(a) $X_\sigma = A_0 \supset A_1 \supset A_2 \supset A_3 \ldots$ and $X_\sigma = B_0 \supset B_1 \supset B_2 \supset B_3 \ldots$ .

(b) Both $\bigcap_{n \in \mathbb{N}} A_n$ and $\bigcap_{n \in \mathbb{N}} B_n$ are countably infinite.

(c) $A_n$ and $B_n$ are complete $T$-sections for each $n \in \mathbb{N}$.

(d) For each $n \in \mathbb{N}$ every point in $A_n$ and $B_n$ is recurrent.
Using Corollary 1, \((B_n)\) we obtain Bratteli-Vershik model for left determined substitution of bounded size on countably infinite alphabet.

**Theorem 4.** Let \(\sigma\) be a left determined substitution of bounded size on an infinite alphabet \(A\) and \((X_\sigma, T)\) be the corresponding subshift. Then there exists an ordered generalized–Bratteli diagram \(B = (V, E, \geq)\) and a Vershik map \(\varphi : Y_B \rightarrow Y_B\) such that \((X_\sigma, T)\) is isomorphic to \((Y_B, \varphi)\).
Using Corollary 1, (sets $B_n$) we obtain Bratteli-Vershik model for left determined substitution of bounded size on countably infinite alphabet.

**Theorem 4.** Let $\sigma$ be a left determined substitution of bounded size on an infinite alphabet $A$ and $(X_\sigma, T)$ be the corresponding subshift. Then there exists an ordered generalized–Bratteli diagram $B = (V, E, \geq)$ and a Vershik map $\varphi : Y_B \to Y_B$ such that $(X_\sigma, T)$ is isomorphic to $(Y_B, \varphi)$.

Using Corollary 1, (sets $A_n$) we obtain stationary Bratteli-Vershik model.

**Theorem 5.** Let $\sigma$ be a left determined substitution of bounded size on an infinite alphabet $A$ and $(X_\sigma, T)$ be the corresponding subshift. Then there exists a stationary ordered generalized–Bratteli diagram $\tilde{B} = (\tilde{V}, \tilde{E}, \geq)$ and a Vershik map $\tilde{\varphi} : Y_{\tilde{B}} \to Y_{\tilde{B}}$ such that $(X_\sigma, T)$ is isomorphic to $(Y_{\tilde{B}}, \tilde{\varphi})$. 
Theorem 6. (Generalized PF Theorem, see [K 98]). Suppose $F$ is a countable, non-negative, irreducible, aperiodic matrix. Suppose that $F$ is recurrent. Then there exists a Perron-Frobenius eigenvalue

$$\lambda = \lim_{n \to \infty} \left( f_{ij}^{(n)} \right)^{1/n} > 0$$

such that:

(a) there exist unique strictly positive left $\ell$ and right $r$ eigenvectors corresponding to $\lambda$,

(b) $\ell \cdot r = \sum_i \ell_i r_i < \infty$ if and only if $F$ is positive recurrent.
**Theorem 7.** Let $B = B(F)$ be a stationary generalized-Bratteli diagram such that the incidence matrix $F$, is irreducible, aperiodic and recurrent. Then there exists a tail invariant measure $\mu$ on the path space $Y_B$.

(1) Let $[\bar{e}] = [e_0 e_1 \ldots e_{n-1}]$ denote a cylinder set of length $n$ such that $r(\bar{e}) = v \in V_n$, then we define:

$$\mu([\bar{e}]) = \frac{\ell_v}{\lambda^n}.$$ 

where $\ell$ is the left eigenvector corresponding to Perron value $\lambda$ of $F$. 

**Example**

$$
\begin{align*}
\mu([e_0 e_1]) &= \frac{\ell_v}{\lambda^2} \\
\end{align*}
$$
Theorem 7. Let $B = B(F)$ be a stationary generalized-Bratteli diagram such that the incidence matrix $F$, is irreducible, aperiodic and recurrent. Then there exists a tail invariant measure $\mu$ on the path space $Y_B$.

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where $\ell$ is the left eigenvector corresponding to Perron value $\lambda$ of $F$.

2. The measure $\mu$ is finite if and only if the left eigenvector $\ell = (\ell_v)$ has the property $\sum_v \ell_v < \infty$. 


Since dynamical systems \((X_\sigma, T)\) and \((Y_B, \varphi)\) are isomorphic, we can push the tail-invariant measure to shift-invariant measure.

**Corollary 8.** Let \(\sigma\) be a left determined substitution of bounded size on an infinite alphabet \(\mathcal{A}\) and \((X_\sigma, T)\) be the corresponding subshift. Assume that the countably infinite substitution matrix \(M\) is irreducible, aperiodic, and recurrent.

1. Then there exists a shift-invariant measure \(\nu\) on \(X_\sigma\).

2. Let \(\ell\) be the left eigenvector of \(M\) corresponding to the Perron value \(\lambda\) of \(M\). The measure \(\nu\) is finite if and only if the left eigenvector \(\ell = (\ell_i)\) has the property \(\sum_i \ell_i < \infty\).
Theorem 9. Let $\sigma$ be a left determined substitution of bounded size on an infinite alphabet $\mathcal{A}$ such that the substitution matrix $M$ is irreducible, aperiodic, and recurrent. Then the shift-invariant probability measure $\nu$ (defined in Corollary 7) on $X_\sigma$ is ergodic.

Proof Sketch. We work with the stationary Bratteli-Vershik model $(B, \varphi, \succeq)$ of $(X_\sigma, T)$. Since $\mu$ is probability, we have $\sum_v \ell_v = 1$.

$M$ is irreducible: For all $i, j$ there exists some $n$ such that $m_{ij}^{(n)} > 0$. Moreover, for a fixed state $i$ there exists $k$ such that $m_{ii}^{(n)} > 0$ for all $n \geq k$. 
**Theorem 9.** Let $\sigma$ be a left determined substitution of bounded size on an infinite alphabet $A$ such that the substitution matrix $M$ is irreducible, aperiodic, and recurrent. Then the shift-invariant probability measure $\nu$ (defined in Corollary 7) on $X_\sigma$ is ergodic.

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$M$ is irreducible: For all $i, j$ there exists some $n$ such that $m^{(n)}_{ij} > 0$. Moreover, for a fixed state $i$ there exists $k$ such that $m^{(n)}_{ii} > 0$ for all $n \geq k$.

Let $[\bar{e}]$ be a cylinder set such that $r(\bar{e}) = \nu \in V_n$. We will show that $\mu(\mathcal{R}([\bar{e}])) = 1$, where $\mathcal{R}$ denotes the tail-equivalence relation.
Proof sketch cont.

For \( n > 0 \) and \( v \in V_n \) we define, \( X_v(n) = \{ x \in Y_B : r(x_{n-1}) = v \in V_n \} \).

For a fixed \( k \in \mathbb{N} \) and \( t \) as in the definition of bounded size the set \( V_k = \{ v - kt, ..., v + kt \} \) denotes a set of \( 2kt + 1 \) vertices.

Since \( M \) is irreducible, there exists a level (say \( V_{i_k} \), \( i_k > n \)) such that \( v \) is connected to every \( u \in V \) on level \( V_{i_k} \).
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Since \( M \) is irreducible, there exists a level (say \( V_{i_k}, i_k > n \)) such that \( v \) is connected to every \( u \in V \) on level \( V_{i_k} \).

Consider the set \( \overline{U_k} = \bigcup_{u \in V_k} X_u(i_k) \subset \mathcal{R}([\bar{e}]). \)

Note that \( \mu(\overline{U_k}) = \sum_{u \in V_k} \frac{|E(V_0, u)|}{\chi_{i_k}} \ell_u. \)

\[
\mu(\overline{U_k}) = \sum_{u \in V_k} \frac{\sum_{w = u - i_k t}^{u + i_k t} f_{uw}(i_k)}{\lambda_{i_k}} \ell_u
\]
Proof sketch cont.

By switching the order of summations

\[ \mu(U_k) = \sum_{w=u-i_k t}^{u+i_k t} \sum_{u \in V_k} (\ell_u) \cdot (f_{uw}^{(i_k)}) \]

For \( \epsilon > 0 \), there exists \( k \in \mathbb{N} \) such that

\[ \sum_{u \in V_k} (\ell_u) \cdot (f_{uw}^{(i_k)}) > \lambda i_k \ell_w - \epsilon \]

Hence, \( \mu(U_k) > \frac{u+i_k t}{w=u-i_k t} \frac{\lambda i_k \ell_w - \epsilon}{\lambda i_k} = \frac{u+i_k t}{w=u-i_k t} \ell_w - \frac{u+i_k t}{w=u-i_k t} \frac{\epsilon}{\lambda i_k} \)
Proof sketch cont.

\[
\mu(U_k) > \sum_{w=u-i_k t}^{u+i_k t} \ell_w - \frac{\epsilon}{\lambda i_k} (2i_k t + 1),
\]

\[k \to \infty, i_k \to \infty,\] observe that

\[
\sum_{w=u-i_k t}^{u+i_k t} \ell_w \to \sum_{w \in \mathbb{Z}} \ell_w = 1 \quad \text{and} \quad \frac{(2i_k t + 1)}{\lambda i_k} \to 0,
\]

Thus \(\mu(U_k) \to 1\). Since \(\mu(U_k) \subset \mathcal{R}([\bar{e}])\), we get \(\mu(\mathcal{R}([\bar{e}])) = 1\).

Thus for any two open sets \(O_1, O_2 \in Y_B\), there exists \(n \in \mathbb{Z}\) such that

\[\mu(\varphi^n_B(O_1) \cap O_2) > 0 \implies \]

For any two Borel sets \(A_1, A_2\), there exists \(n \in \mathbb{Z}\) such that

\[\mu(\varphi^n_B(A_1) \cap A_2) > 0 \implies \mu \text{ is ergodic.} \]
Let $\sigma$ be a left determined substitution of bounded size on an infinite alphabet $\mathcal{A}$ such that the substitution matrix $M$, is irreducible and recurrent, then following could be said about the shift-invariant measure $\nu$ defined earlier.

<table>
<thead>
<tr>
<th>Substitution matrix</th>
<th>$\sum_v \ell_v$</th>
<th>Shift-invariant measure</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Type</td>
</tr>
<tr>
<td>Positive recurrent</td>
<td>Finite</td>
<td>Finite</td>
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<td>Null recurrent</td>
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<td>Positive recurrent</td>
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<td>$\sigma$-finite</td>
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<tr>
<td>Null recurrent</td>
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Example

One step fwd, two step back substitution on $\mathbb{Z}$. Define $\sigma$ by

$$
-1 \mapsto -2 \quad -1 0 ; \quad 0 \mapsto -1 \quad 0 \quad 1
$$

$$
n \mapsto (n - 1)(n + 1)(n + 1) ; \quad n \leq -2
$$

$$
n \mapsto (n - 1)(n - 1)(n + 1) ; \quad n \geq 1
$$

This substitution is left determined, with irreducible, positive recurrent matrix; $\lambda = 3$; left e.v, $\ell = (\ldots \frac{1}{2^4}, \frac{1}{2^3}, \frac{1}{2^2}, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\ldots)$


Thank You
Some definitions from theory of Stochastic Process

- Countably infinite non-negative matrix $F$ is called \textit{irreducible} if for all $i, j \in \mathbb{Z}$ there exists some $n \in \mathbb{N}$ such that $f_{ij}(n) > 0$.

- An irreducible matrix $F$ has period $p$ if, for all vertices $i \in \mathbb{Z}$, $p = \gcd\{\ell : f_{ii}(\ell) > 0\}$. If $p = 1$, the matrix $F$ is called \textit{aperiodic}.

- An irreducible aperiodic matrix $F$ admits a \textit{Perron-Frobenius eigenvalue} $\lambda$, defined by, $\lambda = \lim_{n \to \infty} (f_{ii}(n))^{\frac{1}{n}}$ (independent of $i$).

- An irreducible aperiodic matrix $F$ is called \textit{transient} if $\sum_n f_{ij}(n) \lambda^{-n} < \infty$; otherwise, $F$ is called \textit{recurrent}.

- For a recurrent matrix $F$, define $\ell_{ij}(1) = f_{ij}$ and $\ell_{ij}(n + 1) = \sum_{k \neq i} \ell_{ik}(n) f_{kj}$.

- The matrix $F$ is called \textit{null-recurrent} if $\sum_n n \ell_{ii}(n) \lambda^{-n} < \infty$; otherwise $F$ is called \textit{positive recurrent}.