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Math 592:
Descriptive Set Theory

ASSIGNMENT 1

Due: Sep 16, 23:59

ORDINALS. An **ordinal** is a transitive set well-ordered by membership $\in$: transitivity means that every element of the set is also a subset of it, and the ordering is given by $\alpha < \beta$ iff $\alpha \in \beta$. Every element of an ordinal is itself an ordinal, and each ordinal consists exactly of all the ordinals smaller than it. For example, $0 = \varnothing$, $1 = \{0\}$, $2 = \{0, 1\}$, and, in general, $n = \{0, \ldots, n-1\}$. The first infinite ordinal is $\omega = \mathbb{N} = \{0, 1, 2, \ldots\}$. The **successor** of an ordinal α is $\alpha + 1 = \alpha \cup \{\alpha\}$, the smallest ordinal greater than α. A **limit ordinal** is a nonzero ordinal that is not a successor; equivalently, it is a nonempty ordinal with no greatest element, such as ω. Membership also well-orders the entire class of ordinals, extending the usual ordering of $\mathbb{N}$. Thus a sequence $(A_\alpha)_{\alpha<\gamma}$ is indexed by the elements of the well-ordered set γ. An ordinal is **countable** if it is countable as a set; the first uncountable ordinal, denoted ω_1, is the set of all countable ordinals.

1. Let X be a second-countable topological space.

(a) Encode every open set in X as a subset of $\mathbb{N}$. Deduce that X has at most continuum-many open subsets.

(b) Let α, β, γ denote ordinals. A sequence of sets $(A_\alpha)_{\alpha<\gamma}$ is called *monotone* if it is either increasing (i.e. $\alpha < \beta \Rightarrow A_\alpha \subseteq A_\beta$, for all $\alpha, \beta < \gamma$) or decreasing (i.e. $\alpha < \beta \Rightarrow A_\alpha \supseteq A_\beta$, for all $\alpha, \beta < \gamma$); call it *strictly monotone*, if all of the inclusions are strict.

Prove that any strictly monotone sequence $(U_\alpha)_{\alpha<\gamma}$ of open subsets of X has countable length, i.e. γ is countable.

(c) Show that every monotone sequence $(U_\alpha)_{\alpha<\omega_1}$ of open subsets of X eventually stabilizes, i.e. there is $\beta < \omega_1$ such that for all $\beta \leq \alpha < \omega_1$, we have $U_\alpha = U_\beta$.

Hint. ω_1 is not a supremum of a countable set of countable ordinals.

(d) Conclude the analogues of parts (a), (b) and (c) for closed sets.

2. (a) Show that a metric space X is complete if and only if every decreasing sequence of closed sets $(B_n)_{n\in\mathbb{N}}$ with $\operatorname{diam}(B_n) \to 0$ has nonempty intersection (in fact, $\bigcap_{n\in\mathbb{N}} B_n$ is a singleton).

(b) Show that the requirement in (a) that $\operatorname{diam}(B_n) \to 0$ cannot be dropped. Do this by constructing a complete metric space that has a decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of closed **balls** with $\bigcap_{n\in\mathbb{N}} B_n = \emptyset$.

Hint. Use $\mathbb{N}$ as the underlying set for your metric space.

3. By definition, the class of G_δ sets is closed under countable intersections. Show that it is also closed under finite unions. Equivalently, the class of F_σ sets is closed under finite intersections.

Hint. Think in terms of quantifiers $\forall$ and $\exists$ rather than intersections and unions; for example, if $A = \bigcap_n U_n$, then $x \in A \iff \forall n(x \in A_n)$.

4. Prove that every Polish space X admits a linear ordering $<$ that is both G_δ and F_σ as a subset of X^2.

5. (a) Show that the Cantor set (with relative topology of $\mathbb{R}$) is homeomorphic to the Cantor space.

(b) Show that the Baire space $\mathbb{N}^\mathbb{N}$ is homeomorphic to a G_δ subset of the Cantor space $2^\mathbb{N}$.

(c) [*Optional*] Show that the set of irrationals (with the relative topology of $\mathbb{R}$) is homeomorphic to the Baire space.

Hint. Use the continued fraction expansion.

6. Let $T \subseteq A^{<\mathbb{N}}$ be a tree. Prove that $[T]$ is compact if and only if T is finitely branching.

7. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree. Define a total ordering $<$ on T such that $<$ is a well-ordering if and only if T does not have an infinite branch.

MORE QUESTIONS TO BE ADDED.