Last update: 12:23am, Apr 9.

Measure theory with ergodic horizons | HOMEWORK 5 | **Due: Mar 25**

Definition. Let X be a set. We say that a subset $F \subseteq X$ **separates** $x, y \in X$ if it contains exactly one of x, y. We also say that a family $\mathcal{F}$ of subsets of X **separates points** (in X) if any two distinct points $x, y \in X$ are separated by some $F \in \mathcal{F}$.

1. Let X be a set and $\mathcal{F}$ be a family of subsets of X. Let $E_{\mathcal{F}}$ be the binary relation on X of not being separated by any set in $\mathcal{F}$, i.e. $xE_{\mathcal{F}}y$ if and only if $x \in F \Leftrightarrow y \in F$ for every $F \in \mathcal{F}$.

(a) Observe that $E_{\mathcal{F}}$ is an equivalence relation, and that $\mathcal{F}$ separates points in X exactly when $E_{\mathcal{F}}$ is the equality relation on X (i.e. every $E_{\mathcal{F}}$-class is a singleton).

(b) Prove that every set $B \in \langle \mathcal{F} \rangle_\sigma$ is $E_{\mathcal{F}}$-**invariant**, i.e. is a union of $E_{\mathcal{F}}$-classes.

HINT: Consider the collection of sets in $\langle \mathcal{F} \rangle_\sigma$ that are $E_{\mathcal{F}}$-invariant and show that it is a σ-algebra.

(c) Conclude that if $\mathcal{B}$ is a σ-algebra on X containing all singletons (i.e. $\{x\} \in \mathcal{B}$ for each $x \in X$) then every generating family $\mathcal{F} \subseteq \mathcal{B}$ for $\mathcal{B}$ (i.e. $\langle \mathcal{F} \rangle_\sigma = \mathcal{B}$) separates points in X.

2. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space.

(a) Prove that if $\mathcal{B}$ contains a countable family $\mathcal{F}$ separating points in X, then every μ-atom A is $=_\mu$ to a singleton, i.e. $A = \{x\} \cup Z$ where $\{x\}$ has positive measure and Z is null. In particular, every Borel measure on a second countable (equivalently, separable) metric space has this property.

HINT: Note that it is enough to prove this for a finite measure μ. Let $(F_n)_{n \in \mathbb{N}}$ be an enumeration of $\mathcal{F}$ and build a so-called **Cantor scheme** on X by setting $F_s := \bigcap_{i<|s|} F_i^{s(n)}$ for each $s \in 2^{<\mathbb{N}}$, where we use notation $B^0 := X \setminus B$ and $B^1 := B$ for a set $B \subseteq X$. Now if A is an atom, note that for each $n \in \mathbb{N}$ there is exactly one $s \in 2^n$ with $\mu(F_s \cap A) = \mu(A)$. Follow this down in the tree $2^{<\mathbb{N}}$ to find an infinite branch, which defines a singleton of measure $\mu(A)$. (Just like in the proofs that $2^{\mathbb{N}}$ and $[0, 1]$ are sequentially compact.)

(b) Conclude that if $\mathcal{B}$ contains all singletons and is countably generated (i.e. admits a countable generating subcollection), then every μ-atom is $=_\mu$ to a singleton.

3. Finish the proof of the statement that locally finite Borel measures on Polish spaces are tight. You may use without proof that every open subset of a Polish metric space is still Polish but perhaps with respect to a different equivalent metric.