Measure theory with ergodic horizons

Homework 14

Due: Jun 17

1. Prove that for every monotone function $f : \mathbb{R} \to \mathbb{R}$, its set C_f of continuity points is cocountable (i.e. f is continuous at every point $x \in \mathbb{R} \setminus Q$ for some countable Q) and f' exists a.e.

Instructions: For the existence of f' a.e. you may use the fact from lecture that this is true for increasing right-continuous functions.

Definition. For a function $f : \mathbb{R} \to \mathbb{R}$, define its **variation function** $T_f : \mathbb{R} \to [0, \infty]$ by

$$T_f(x) := \sup \left\{ \sum_{i<n} |f(x_{i+1}) - f(x_i)| : n \in \mathbb{N} \text{ and } x_0 < x_1 < \cdots < x_n \leqslant x \right\}.$$

Say that f has **bounded total variation** if $T_f(\infty) := \lim_{x \to \infty} T_f(x) < \infty$.

2. Let $f : \mathbb{R} \to \mathbb{R}$.

(a) Prove that T_f is increasing, and if f is right-continuous then T_f is right-continuous.

(b) Prove that $T_f + f$ and $T_f - f$ are increasing.

Hint: To show that a function $g : \mathbb{R} \to \overline{\mathbb{R}}$ is increasing, you need to show that $g(b) - g(a) \geqslant 0$ for all $a < b$.

(c) Conclude that f has bounded variation if and only if f is a difference of two bounded increasing functions.

(d) Deduce that f is a distribution of a (unique) finite Borel signed measure ν if and only if f is right-continuous and has bounded variation.

3. Prove that for a function $f : \mathbb{R} \to \mathbb{R}$, the following are equivalent:

(1) f is a distribution of a (unique) finite Borel signed measure $\nu \ll \lambda$.

(2) f' exists a.e. and is in $L^1(\lambda)$, and the fundamental theorem of calculus holds: for each $a < b$,

$$f(b) - f(a) = \int_a^b f' \, d\lambda.$$

(3) f has bounded variation and is absolutely continuous.

Instructions: Only sketch the proofs without details.