Last update: 7:13pm, May 20.

Measure theory with ergodic horizons | HOMEWORK 12 | **Due: May 27**

1. Let (Y,ν) be a probability space and consider the product space $(X,\mu) := (Y^{\mathbb{N}},\mu)$. Let $S : X \to X$ be the shift transformation, i.e. $(y_n)_{n\in\mathbb{N}} \mapsto (y_{n+1})_{n\in\mathbb{N}}$. Prove:

(a) S is measure-preserving, i.e. $\mu(S^{-1}(A)) = \mu(A)$ for each measurable $A \subseteq X$.

(b) S is mixing, i.e. for all measurable $A, B \subseteq X$, we have

$$\lim_{n\in\infty} \mu(S^{-1}(A)\cap B) = \mu(A)\mu(B).$$

HINT: Fist prove for cylinders A, B and then approximate.

(c) For $k \geqslant 2$, the k-fold bakers map $b_k : [0,1) \to [0,1)$ with Lebesgue measure λ on $[0,1)$ is measure-isomorphic to the shift transformation on $(k^{\mathbb{N}}, \nu_u^{\mathbb{N}})$, where ν_u is the uniform probability measure on $k := \{0,1,\dots,k-1\}$.

2. Proof of the classical ergodic theorem without the simplifying assumptions. The following steps remove the assumptions that the functions f and $x \mapsto n_x$ are bounded.

(a) Let $\delta > 0$ be small enough so that for each measurable set $B \subseteq X$

$$\mu(B) \leqslant \delta \implies \int_B |f-\delta| d\mu \leqslant \frac{\Delta}{4}.$$

(b) Let $M > 0$ be large enough so that $Y := f^{-1}([-M,\infty))$ has measure $\geqslant 1-\delta$. Thus, $\mathbb{1}_Y(f-\Delta) \geqslant -(M+\Delta)$ and $\int \mathbb{1}_{X\setminus Y}|f-\Delta| d\mu \leqslant \frac{\Delta}{4}$.

(c) Let $L \in \mathbb{N}$ be large enough so that the set

$$Z := \{x \in X : n_x > L\}$$

has measure $\leqslant \varepsilon\cdot\delta$, where $\varepsilon := \frac{1}{2(M+\Delta)}\frac{\Delta}{4}$. Thus, by (b) of the local-global bridge (small[2] measure implies small density), for every $n \in \mathbb{N}$, we have $A_n \mathbb{1}_Z(x) = \frac{|I_n(x)\cap Z|}{|I_n(x)|} \leqslant \varepsilon$ for all x in a set $X_n \subseteq X$ of measure $\geqslant 1-\delta$.

(d) **Tiling.** Let $N \in \mathbb{N}$ be large enough so that $\frac{L}{N} < \varepsilon$. Then for each $x \in X_N$, at least $(1-2\varepsilon)$-fraction of the set $I_N(x)$ is tiled by intervals of the form $I_{n_y}(y)$. Thus, $A_N(\mathbb{1}_Y(f-\Delta))(x) \geqslant -2\varepsilon(M+\Delta) = -\frac{\Delta}{4}$.

HINT: Do not tile the points of Z in $I_N(x)$. They occupy at most ε-fraction of $I_N(x)$.

(e) **Local-global contradiction.** $-\Delta = \int (f-\Delta)d\mu \geqslant \int \mathbb{1}_Y(f-\Delta)d\mu - \frac{\Delta}{4}$, and the local-global bridge (again!) gives:

$$\int \mathbb{1}_Y(f-\Delta)d\mu = \int A_N(\mathbb{1}_Y(f-\Delta))d\mu \geqslant \int_{X_N} A_N(\mathbb{1}_Y(f-\Delta))d\mu - \frac{\Delta}{4} \geqslant -\frac{\Delta}{4} - \frac{\Delta}{4},$$

so $-\Delta \geqslant -\frac{\Delta}{4} - \frac{\Delta}{4} - \frac{\Delta}{4} = -\frac{3}{4}\Delta$, a contradiction.

MORE QUESTIONS TO BE ADDED.