Last update: 5:50am, May 1.

Measure theory with ergodic horizons | Homework 10 | **Due: May 6**

0. [*Optional*] **Riemann integration.** Let λ be the Lebesgue measure on $\mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ be a bounded function, $a < b \in \mathbb{R}$. For a finite partition $\mathcal{P}$ of $[a,b]$ into intervals, let $\|\mathcal{P}\|$ denote its **mesh**, i.e. maximum length of an interval in $\mathcal{P}$. Let $\underline{f}_{\mathcal{P}} := \sum_{I \in \mathcal{P}} a_I \mathbb{1}_I$ and $\overline{f}_{\mathcal{P}} := \sum_{I \in \mathcal{P}} A_I \mathbb{1}_I$, where $a_I := \inf_{x \in I} f(x)$ and $A_I := \sup_{x \in I} f(x)$. Fix a sequence $(\mathcal{P}_n)$ of finite partitions of $[a,b]$ into intervals such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \in \mathbb{N}$, and $\|\mathcal{P}_n\| \to 0$ as $n \to \infty$.

(a) Prove that the sequences $(\underline{f}_{\mathcal{P}_n})$ and $(\overline{f}_{\mathcal{P}_n})$ are monotone, hence the limits $\underline{f} := \lim_n \underline{f}_{\mathcal{P}_n}$ and $\overline{f} := \lim_n \overline{f}_{\mathcal{P}_n}$ exist and are Borel functions such that $\underline{f} \leqslant f \leqslant \overline{f}$.

(b) Recall the definition of a Riemann integrable function, and prove that f is Reimann integrable if and only if $\int \underline{f}\, d\lambda = \int \overline{f}\, d\lambda$ if and only if $\underline{f} = \overline{f}$ a.e.

Hint: For the first equivalence, note that $\int \underline{f}\, d\lambda$ and $\int \overline{f}\, d\lambda$ are exactly the limits of the lower and upper sums of the partition $\mathcal{P}_n$.

(c) Deduce that if f is Riemann integrable then it is Lebesgue measurable and its Riemann integral $\int_a^b f(t)dt$ is equal to its Lebesgue integral $\int_{[a,b]} f\, d\lambda$.

(d) Also prove that f is Riemann integrable if and only if it is continuous at a.e. point in $[a,b]$, i.e. the set C_f of continuity points of f is conull in $[a,b]$.

Hint: This question is partially answered in Folland's "Real Analysis", Theorem 2.28 on page 57.

1. Let $f_n \in L^1(\mathbb{R}, \lambda)$ be a non-negative Lebesgue integrable functions on $\mathbb{R}$. Prove or give a counterexample to the following statements.

(a) $\displaystyle\int \limsup_{n\to\infty} f_n \geqslant \limsup_{n\to\infty} \int f_n.$

(b) If $f_n \to 0$ both pointwise and in the L^1-norm, then there is $g \in L^1(\mathbb{R}, \lambda)$ such that $f_n \leqslant g$ for each $n \in \mathbb{N}$.

2. Prove the **generalized dominated convergence theorem**: Let (X, μ) be a measure space and f_n, f be μ-measurable functions be such that $f_n \to f$ a.e. If there are non-negative $g_n, g \in L^1$ such that $g_n \to g$ a.e., $\int g_n\, d\mu \to \int g\, d\mu$, and $|f_n| \leqslant g_n$ for each $n \in \mathbb{N}$, then $f_n \to_{L^1} f$. In particular, $\int f_n\, d\mu \to \int f\, d\mu$.

3. Let $f_n, f \in L^1$ be such that $f_n \to f$ a.e. and $\int |f_n| \to \int |f|$.

(a) Prove that $f_n \to_{L^1} f$.

(b) Conclude that $\int_A f_n \to \int_A f$ for each measurable $A \subseteq X$.

4. Consider $\mathbb{R}^d$ with Lebesgue measure λ and let $L^1 := L^1(\mathbb{R}^d, \lambda)$.

(a) Prove that for every $f \in L^1$ and $\varepsilon > 0$, there is a simple function s that is a linear combination of indicator functions of bounded boxes such that $\|f - s\|_1 < \varepsilon$.

Hint: Firstly, make things bounded by noting that $\|f - f\mathbb{1}_{B_N}\|_1 < \varepsilon/2$ for all large enough $N \in \mathbb{N}$, where B_N is the cube of side-length N centered at the origin.

(b) Prove that for every bounded box $B \subseteq \mathbb{R}^d$ and $\varepsilon > 0$, there is a continuous function $g_B : \mathbb{R}^d \to \mathbb{R}$ with support $\subseteq B$ such that $\|\mathbb{1}_B - g_B\|_1 < \varepsilon$.

Hint: Do this for $d = 1$ first.

(c) Deduce that for every $f \in L^1$ and $\varepsilon > 0$, there is a continuous function $g : \mathbb{R}^d \to \mathbb{R}$ of bounded support such that $\|f - g\|_1 < \varepsilon$. In other words, continuous functions (of bounded support) are dense in L^1.

MORE QUESTIONS TO BE ADDED.