

Math 564: Real analysis & measure theory**HOMEWORK 4****Due: Nov 4, 11:59pm**

Definition. Let X, Y be topological/metric spaces. A set $M \subseteq X$ is called **universally measurable** if it is μ -measurable for *every* σ -finite Borel measure μ on X . It is immediate to check that the universally measurable sets form a σ -algebra, which we denote by $\text{UM}(X)$.

A function $f : X \rightarrow Y$ is called **universally measurable** if it is $\text{UM}(X)$ -measurable.

Remark. By definition, all Borel sets and functions are universally measurable, but are there non-Borel universally measurable sets? It is a theorem of descriptive set theory that there are. In fact all continuous images (e.g. projections) of Borel sets (called **analytic sets**) are universally measurable and some are not Borel.

1. Let X, Y, Z be topological/metric spaces. Prove:

- (a) A set $M \subseteq X$ is universally measurable if and only if it is μ -measurable for every Borel *probability* measure μ on X .
- (b) Universally measurable functions are closed under compositions. More precisely, if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are universally measurable, then $g \circ f : X \rightarrow Z$ is universally measurable.

REMARK: This is in contrast with a previous homework question exhibiting a Lebesgue measurable g and Borel f such that $g \circ f$ is not Lebesgue measurable.

Definition. Let $(X_i, \mathcal{I}_i)$, $i \in \{0, 1\}$ be measurable spaces. Note that the projection $\text{proj}_i : X_0 \times X_1 \rightarrow X_i$ is $(\mathcal{I}_0 \otimes \mathcal{I}_1, \mathcal{I}_i)$ -measurable.

The **marginals** of a measure μ on $(X_0 \times X_1, \mathcal{I}_0 \otimes \mathcal{I}_1)$ are the pushforward measures $(\text{proj}_0)_*\mu$ and $(\text{proj}_1)_*\mu$ under projections. Conversely, a **coupling** of measures μ_i on $(X_i, \mathcal{I}_i)$ is a measure μ on $(X_0 \times X_1, \mathcal{I}_0 \otimes \mathcal{I}_1)$ whose marginals are μ_0 and μ_1 , i.e. $(\text{proj}_i)_*\mu = \mu_i$ for all $i \in \{0, 1\}$.

2. Find a coupling of the Bernoulli measures $\mu_{2/5}$ and $\mu_{1/3}$ on $2^{\mathbb{N}}$.

HINT: Identify $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ with $(2 \times 2)^{\mathbb{N}}$.

3. **Poincaré recurrence theorem.** Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \rightarrow X$ be a $(\mathcal{B}, \mathcal{B})$ -measurable μ -preserving transformation, i.e. $\mu(T^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{B}$. Call a set $W \subseteq X$ **T -wandering** if the preimages $T^{-n}(W)$ are pairwise disjoint for $n \in \mathbb{N}$. Call a set $B \subseteq X$ **T -recurrent** if for a.e. $x \in B$, there are infinitely many $n \in \mathbb{N}$ such that $T^n(x) \in B$. (Note that μ -null sets are (vacuously) T -recurrent.)

- (a) Observe that T is $(\text{Meas}_\mu, \text{Meas}_\mu)$ -measurable.
- (b) Show that every μ -measurable T -wandering set $W \subseteq X$ is μ -null.
- (c) Deduce the Poincaré recurrence theorem, which says that every μ -measurable set $B \subseteq X$ is T -recurrent.

HINT: Consider the set $W := \{x \in B : \forall n \geq 1 \ T^n(x) \notin B\}$.

4. Let $(X, \mathcal{B}, \mu)$ be a measure space. Prove that the integral of simple functions is well-defined, i.e. for all μ -measurable sets $A_i, B_j \subseteq X$ and $a_i, b_j \in \mathbb{R}$,

$$\sum_{i < n} a_i \mathbb{1}_{A_i} = \sum_{j < m} b_j \mathbb{1}_{B_j} \text{ implies } \sum_{i < n} a_i \mu(A_i) = \sum_{j < m} b_j \mu(B_j).$$

5. For a measure space (X, μ) , we call a non-negative function $f : X \rightarrow [0, \infty]$ **μ -integrable** if it is μ -measurable and $\int f d\mu < \infty$.

Let (f_n) be a sequence of non-negative λ -integrable functions $\mathbb{R} \rightarrow \mathbb{R}$, where λ denotes the Lebesgue measure. Prove or give a counterexample to the following statements.

- (a) $\int \limsup_{n \rightarrow \infty} f_n \geq \limsup_{n \rightarrow \infty} \int f_n$.
- (b) If $\lim_{n \rightarrow \infty} f_n = 0$ pointwise and $\lim_{n \rightarrow \infty} \int f_n d\lambda = 0$, then there is an integrable dominating function for (f_n) , i.e. a non-negative λ -integrable function $g : \mathbb{R} \rightarrow [0, \infty]$ such that $f_n \leq g$ for each $n \in \mathbb{N}$.

Definition. Let $a < b \in \mathbb{R}$ and $f : [a, b] \rightarrow \mathbb{R}$ be an arbitrary function. For a finite partition $\mathcal{P}$ of $[a, b]$ into intervals, the quantity

$$\|\mathcal{P}\| := \max_{I \in \mathcal{P}} \text{lh}(I),$$

where $\text{lh}(I)$ is the length of the interval I , is called the **mesh** of $\mathcal{P}$. Let $L(f, \mathcal{P}) := \sum_{I \in \mathcal{P}} \ell_I \text{lh}(I)$ and $U(f, \mathcal{P}) := \sum_{I \in \mathcal{P}} u_I \text{lh}(I)$, where $\ell_I := \inf_{x \in I} f(x)$ and $u_I := \sup_{x \in I} f(x)$, and where we adopt the convention that $\alpha \cdot \infty := \infty$ and $\alpha \cdot (-\infty) := -\infty$ for all $\alpha > 0$. Define

$$L_a^b f := \sup_{\mathcal{P}} L(f, \mathcal{P}) \quad \text{and} \quad U_a^b f := \inf_{\mathcal{P}} U(f, \mathcal{P}),$$

where $\mathcal{P}$ ranges over all finite partitions of $[a, b]$ into intervals. Call f **Riemann integrable** if $L_a^b f = U_a^b f$, and call this common value the **Riemann integral** of f from a to b , denoted $\int_a^b f dx$.

6. Let $a < b \in \mathbb{R}$ and $f : [a, b] \rightarrow \mathbb{R}$ be an arbitrary function.

- (a) [Optional] Note that if f is unbounded above (resp. below) at a point $x \in [a, b]$ (i.e. f is unbounded above/below in every neighbourhood of x) then $\bar{f}_{\mathcal{P}}(x) = \infty$ (resp. $\underline{f}_{\mathcal{P}}(x) = -\infty$). Deduce that if f is Riemann integrable then it is bounded.
- (b) Observe that for partitions $\mathcal{P}_0$ and $\mathcal{P}_1$ of $[a, b]$, if $\mathcal{P}_1$ refines $\mathcal{P}_0$ (we write $\mathcal{P}_0 \leq \mathcal{P}_1$), then $L(f, \mathcal{P}_0) \leq L(f, \mathcal{P}_1)$ and $U(f, \mathcal{P}_1) \leq U(f, \mathcal{P}_0)$. Use this to show that there is sequence $\mathcal{P}_0 \leq \mathcal{P}_1 \leq \dots$ of refining partitions of $[a, b]$ into intervals such that $\lim_{n \rightarrow \infty} \|\mathcal{P}_n\| = 0$ and

$$L_a^b f = \lim_{n \rightarrow \infty} L(f, \mathcal{P}_n) \quad \text{and} \quad U_a^b f = \lim_{n \rightarrow \infty} U(f, \mathcal{P}_n).$$

- (c) For a finite partition $\mathcal{P}$ of $[a, b]$ into intervals, let $\ell(f, \mathcal{P}) := \sum_{I \in \mathcal{P}} \ell_I \mathbb{1}_I$ and $u(f, \mathcal{P}) := \sum_{I \in \mathcal{P}} u_I \mathbb{1}_I$, where $\ell_I := \inf_{x \in I} f(x)$ and $u_I := \sup_{x \in I} f(x)$. In particular, $\ell(f, \mathcal{P})$ and $u(f, \mathcal{P})$ are Borel simple functions, and $L(f, \mathcal{P}) = \int \ell(f, \mathcal{P}) d\lambda$ and $U(f, \mathcal{P}) = \int u(f, \mathcal{P}) d\lambda$, where λ is the Lebesgue measure on $\mathbb{R}$.

Prove that if f is Riemann integrable, then f is Lebesgue measurable and its Lebesgue integral coincides with the Riemann integral.

HINT: Fix any sequence $(\mathcal{P}_n)$ as in part (b) and prove:

$$\lim_{n \rightarrow \infty} \ell(f, \mathcal{P}_n) = f = \lim_{n \rightarrow \infty} u(f, \mathcal{P}_n) \text{ a.e.}$$

Use dominated convergence theorem along the way and afterwards.

- (d) Prove that f is Riemann integrable if and only if f is bounded and continuous a.e. (i.e. continuous at a.e. point of $[a, b]$).
7. Prove that for a countably generated¹ measure space $(X, \mathcal{B}, \mu)$, there is a countable collection of simple functions which is dense in $L^1(X, \mu)$ in the L^1 -metric. In particular, $L^1(X, \mu)$ is separable.

MORE QUESTIONS TO BE ADDED.

¹The definition is at the end of Lecture 15.