Math 564: Real analysis & measure theory | HOMEWORK 3 | **Due: Oct 7, 11:59pm**

Definition. Let X be a set. We say that a subset $F \subseteq X$ **separates** $x, y \in X$ if it contains exactly one of x, y. We also say that a family $\mathcal{F}$ of subsets of X **separates points** (in X) if any two distinct points $x, y \in X$ are separated by some $F \in \mathcal{F}$.

1. Let X be a set and $\mathcal{F}$ be a family of subsets of X. Let $E_{\mathcal{F}}$ be the binary relation on X of not being separated by any set in $\mathcal{F}$, i.e. $xE_{\mathcal{F}}y$ if and only if $x \in F \Leftrightarrow y \in F$ for every $F \in \mathcal{F}$.

(a) Observe that $E_{\mathcal{F}}$ is an equivalence relation, and that $\mathcal{F}$ separates points in X exactly when $E_{\mathcal{F}}$ is the equality relation on X (i.e. every $E_{\mathcal{F}}$-class is a singleton).

(b) Prove that every set $B \in \langle \mathcal{F} \rangle_\sigma$ is $E_{\mathcal{F}}$**-invariant**, i.e. is a union of $E_{\mathcal{F}}$-classes.

HINT: Consider the collection of sets in $\langle \mathcal{F} \rangle_\sigma$ that are $E_{\mathcal{F}}$-invariant and show that it is a σ-algebra.

(c) Conclude that if $\mathcal{B}$ is a σ-algebra on X containing all singletons (i.e. $\{x\} \in \mathcal{B}$ for each $x \in X$) then every generating family $\mathcal{F}$ for $\mathcal{B}$ (i.e. $\langle \mathcal{F} \rangle_\sigma = \mathcal{B}$) separates points in X.

2. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space.

(a) Prove that if $\mathcal{B}$ contains a countable family $\mathcal{F}$ separating points in X, then every μ-atom A is $=_\mu$ to a singleton, i.e. $A = \{x\} \cup Z$ where $\{x\}$ has positive measure and Z is null.

HINT: Firstly, note that σ-finiteness implies $\mu(A) < \infty$. Next, define a decreasing sequence (A_n) of subsets of A as follows: put $A_0 := A$ and supposing that A_n is defined, let A_{n+1} be the unique non-null set among $F_n \cap A_n$ or $F_n^c \cap A_n$. Finally, show that $\bigcap_{n \in \mathbb{N}} A_n$ is a singleton.

(b) Conclude that if $\mathcal{B}$ contains all singletons and is countably generated (i.e. admits a countable generating subcollection), then every μ-atom is $=_\mu$ to a singleton.

3. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space.

(a) Prove that there are at most countably many disjoint atoms in $\mathcal{B}$.

(b) From this and Question 2(b) deduce that every purely atomic σ-finite Borel measure on a second countable metric space X is a positive linear combination of Dirac measures, i.e.

$$\mu = \sum_{n \in \mathbb{N}} a_n \delta_{x_n},$$

for some $a_n \geqslant 0$ and $x_n \in X$.

(c) Conclude the following decomposition theorem:

Theorem. *Every σ-finite Borel measure μ on a second countable metric space X decomposes into purely atomic and atomless parts, i.e. $\mu = \mu_0 + \mu_1$, where μ_0 is an atomless Borel measure on X and μ_1 is a purely atomic Borel measure on X. Equivalently, $X = X_0 \sqcup X_1$ where X_1 is countable, $\mu|_{X_1}$ is purely atomic, and $\mu|_{X_0}$ is atomless.*

4. Follow the steps below to prove the **Steinhaus theorem**: For every Lebesgue measurable non-null set $A \subseteq \mathbb{R}^d$, the difference set $A - A := \{a_0 - a_1 : a_0, a_1 \in A\}$ contains an open neighbourhood of $\vec{0}$.

Tip: For simplicity of thought and pictures, only think of $d = 1$. Draw pictures.

(i) Check that for all sets $U, V \subseteq \mathbb{R}^d$, we have $U \subseteq V - V$ if and only if $(V + u) \cap V \neq \emptyset$ for all $u \in U$.

(ii) Let B be a nonempty bounded open box whose at least $(1 - 1/2^{d+2}) \cdot 100\%$ is A. Let b_0 be the midpoint of B and put $U := B - b_0$, so U is an open box centered at $\vec{0}$. Show that for each $u \in U$, the intersection $B_u := B + u \cap B$ is a box whose each dimension is at least half of that of B, so B_u occupies at least $(1/2^d) \cdot 100\%$ of B and of $B + u$.

(iii) Conclude that at least 75% of B_u is A while at least 75% of B_u is $A + u$, so $A + u \cap A \neq \emptyset$ for each $u \in U$, hence $U \subseteq A - A$.

Remark: The exact same theorem holds for the Bernoulli(1/2) measure on $2^{\mathbb{N}}$ identified with the group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$. As usual, the proof is easier than for the Lebesgue measure.

5. Recall the equivalence relation $\mathbb{E}_0$ of eventual equality on $2^{\mathbb{N}}$ from Question 8 of HW2. Use the 99% lemma to prove that $\mathbb{E}_0$ is $\mu_{\frac{1}{2}}$-ergodic, where $\mu_{\frac{1}{2}}$ is the Bernoulli($\frac{1}{2}$) measure.

Hint: $\sigma_n(A) = A$ for each $\mathbb{E}_0$-invariant set A and $n \in \mathbb{N}$, where σ_n is the n^{th} bit-flip transformation.

Remark: $\mathbb{E}_0$ is actually μ_p-ergodic for all $p \in (0,1)$ by a more careful version of the same proof.

6. Let Γ be a countable group and $(X, \mathcal{B}, \mu)$ be nonzero atomless measure space. Let $\Gamma \curvearrowright X$ be an action of Γ on X mapping sets in $\mathcal{B}$ to sets in $\mathcal{B}$, i.e. $\gamma \cdot B \in \mathcal{B}$ for each $\gamma \in \Gamma$ and $B \in \mathcal{B}$. Suppose that this action is **null-preserving**, i.e. $\gamma \cdot B$ is null if and only if B is null for each $\gamma \in \Gamma$ and $B \in \mathcal{B}$. Prove that if this action is *ergodic*, then the orbit equivalence relation E_Γ does not admit any μ-measurable transversal.

Hint: For any subset $B \subseteq X$, the set $[B]_{E_\Gamma} := \bigcup_{\gamma \in \Gamma} \gamma \cdot B$ is the smallest E_Γ-invariant set containing B, and is called the E_Γ-**saturation** of B. Now suppose that $S \subseteq X$ is a μ-measurable transversal for E_Γ and use that S is not an atom. To get an E_Γ-invariant set take the E_Γ-saturation.

7. Let (X, d) be a metric space and μ be a regular Borel measure on X.

(a) Prove for every compact subset $K \subseteq X$,

$$\lim_{\delta \searrow 0} \mu(B_\delta(K)) = \mu(K).$$

where $\mu(B_\delta(K))$ denotes the δ-ball around K, i.e.

$$B_\delta(K) := \{x \in X : d(x,k) < \delta\} = \bigcup_{x \in K} B_\delta(x).$$

(b) Construct an example of a non-compact null (in fact, countable) subset of $\mathbb{R}$ for which the conclusion of part (a) fails for Lebesgue measure.

8. (a) Let (X,μ) be a measure space and Y,Z be topological/metric spaces. Show that if $f : X \to Y$ is μ-measurable and $g : Y \to Z$ is Borel then $g \circ f : X \to Z$ is μ-measurable.

(b) The roles of f and g above cannot be switched! Follow the steps below to build an example of a Borel function $f : [0,1] \to [0,1]$ (in fact, a homeomorphism) and a Lebesgue measurable function $g : [0,1] \to [0,1]$ such that the composition $g \circ f$ is **not** Lebesgue measurable.

(i) [*Optional*] Prove that every Lebesgue measurable set A of positive measure contains a non-measurable subset.

Hint: Any transversal of $E_{\mathbb{Q}}|_A$ is non-measurable, and the proof is the same as for $A := [0,1]$ done in class after restricting A to a set of finite positive measure.

(ii) [*Optional*] Let C_0 and C_+ be Cantor sets contained in $(0,1)$ where C_0 is Lebesgue null, while C_+ has positive Lebesgue measure. For convenience, construct C_0 and C_+ by removing middle open intervals at every step. Then there is a homeomorphism $f : [0,1] \to [0,1]$ such that $f(C_+) = C_0$ and $f([0,1] \setminus C_+^c) \subseteq [0,1] \setminus C_0^c$.

(iii) Let $Y \subseteq C_+$ be a non-Lebesgue-measurable set and put $g := \mathbb{1}_{f(Y)}$. Show that g is Lebesgue measurable, however $g \circ f$ is not.

9. Let $(X,\mathcal{A})$ and $(Y_i,\mathcal{B}_i)$, $i = 1,2$, be measurable spaces. Denote by $\mathcal{B}_1 \otimes \mathcal{B}_2$ the σ-algebra on $Y_1 \times Y_2$ generated by the sets of the form $B_1 \times B_2$, where $B_i \in \mathcal{B}_i$.

(a) Prove that if the Y_i are second countable topological/metric spaces, then

$$\mathcal{B}(Y_1 \times Y_2) = \mathcal{B}(Y_1) \otimes \mathcal{B}(Y_2).$$

(Here $Y_1 \times Y_2$ is equipped with the product topology.[1]) In particular, $\mathcal{B}(\mathbb{R}^d) = \bigotimes_{i=1}^d \mathcal{B}(\mathbb{R})$.

(b) Prove that for $(\mathcal{A},\mathcal{B}_i)$-measurable functions $f_i : X \to Y_i$, the function $(f_1,f_2) : X \to Y_1 \times Y_2$ defined by $x \mapsto (f_1(x), f_2(x))$ is $(\mathcal{A},\mathcal{B}_1 \otimes \mathcal{B}_2)$-measurable.

[1] The topology whose open sets are *arbitrary* unions of sets of the form $U_1 \times U_2$, where $U_i \subseteq Y_i$ is open. If the Y_i are metric spaces with the metrics d_i, then the product topology on $Y_1 \times Y_2$ is given by the metric $max(d_1,d_2)$, for example.

(c) Conclude that if $f_1, f_2 : X \to \mathbb{R}$ are $\mathcal{A}$-measurable and $g : \mathbb{R}^2 \to \mathbb{R}$ is Borel, then $g(f_1, f_2) : X \to \mathbb{R}$ is μ-measurable. In particular, $f_1 + f_2$ and $f_1 \cdot f_2$ are μ-measurable.