Last update: 10:38pm, Sept 25.

Math 564: Real analysis & measure theory | Homework 2 | **Due: Sep 23, 11:59pm**

1. Let $\mathcal{A}$ denote the algebra generated by boxes in $\mathbb{R}^d$.
   (a) Prove that $\mathcal{A}$ is exactly the collection of finite disjoint unions of boxes.
   (b) Finish the proof of Claim (b) for boxes in $\mathbb{R}^d$, namely: For any finite partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of a set $A \in \mathcal{A}$ into boxes, we have
   $$\sum_{P_1 \in \mathcal{P}_1} \tilde{\lambda}(P_1) = \sum_{P_2 \in \mathcal{P}_2} \tilde{\lambda}(P_2).$$
   (c) Deduce that λ is finitely additive on $\mathcal{A}$.

2. Let $\mathcal{A}$ denote the algebra generated by boxes in $\mathbb{R}^d$. Using the statement that λ is countably additive on *bounded* boxes, i.e. $\lambda(B) = \sum_{n \in \mathbb{N}} \lambda(B_n)$ for a bounded box $B \subseteq \mathbb{R}^d$ and a partition $\{B_n\}_{n \in \mathbb{N}}$ of B into boxes, finish the proof of countable additivity of λ on $\mathcal{A}$. More precisely:
   (i) Prove that $\lambda(B) = \sum_{n \in \mathbb{N}} \lambda(B_n)$ for an unbounded box $B \subseteq \mathbb{R}^d$ and a partition $\{B_n\}_{n \in \mathbb{N}}$ of B into boxes.

   Caution: An unbounded box has measure ∞ or 0.

   (ii) Finally conclude that $\lambda(A) = \sum_{n \in \mathbb{N}} \lambda(A_n)$ for a set $A \in \mathcal{A}$ and a partition $\{A_n\}_{n \in \mathbb{N}}$ of A into sets in $\mathcal{A}$.

3. *Carathéodory's proof of the existence part of Carathéodory's extension theorem.* Let μ be a premeasure on an algebra $\mathcal{A}$ and let μ^* be its outer measure. For sets $A, B \subseteq X$, say that a set A **conserves** B if $\mu^*(B) = \mu^*(A \cap B) + \mu^*(A^c \cap B)$. Let $\mathcal{M}$ be the collection of **conservative sets**, i.e. sets A which conserve every set $B \subseteq X$. Prove:
   (a) $\mathcal{M} \supseteq \mathcal{A}$.
   (b) $\mathcal{M}$ is an algebra.
   (c) $\mathcal{M}$ is closed under countable disjoint unions. Deduce that $\mathcal{M}$ is a σ-algebra.

   Hint: First show that $\mu^*(S) \geqslant \mu^*(S \cap \bigsqcup_{n<N} M_n) + \mu^*(S \cap M^c)$, then write $\mu^*(S \cap \bigsqcup_{n<N} M_n)$ as a finite sum using part (b) and let $N \to \infty$.

   (d) μ^* is finitely additive on $\mathcal{M}$, and hence countably additive.

4. Let μ be a finite premeasure on an algebra $\mathcal{A}$ on a set X and denote $\mathcal{B} := \langle A \rangle_\sigma$. Recall the pseudo-metric d_{μ^*} on $\mathcal{P}(X)$ defined by $d_{\mu^*}(A, B) := \mu^*(A \triangle B)$. Let $\mathcal{M}_C$ and $\mathcal{M}_T$ be the σ-algebras containing $\mathcal{A}$ given by Carathéodory's and Tao's proofs, respectively, i.e.
   - $\mathcal{M}_C$ is the collection of all sets $M \subseteq X$ such that $\mu^*(S) = \mu^*(M \cap S) + \mu^*(M^c \cap S)$ for all $S \subseteq X$;

- $\mathcal{M}_T$ is the closure of $\mathcal{A}$ with respect to d_{μ^*}.

Prove that $\mathcal{M}_C = \left\{M \subseteq X : d_{\mu^*}(M,B) = 0 \text{ for some } B \in \langle\mathcal{A}\rangle_\sigma\right\} = \mathcal{M}_T$.

Hint: Recall the definition of $\mu^*(M)$ to define B as a countable intersection of countable unions of sets in $\mathcal{A}$. You can use that μ^* is countably additive on $\mathcal{M}_C$ and $\mathcal{M}_T$.

5. A **translation-invariant** Borel measure on $\mathbb{R}^d$ is a measure μ defined on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ such that $\mu(\vec{x}+B) = \mu(B)$ for each point $\vec{x} \in \mathbb{R}^d$ and each Borel set $B \subseteq \mathbb{R}^d$.

(a) Show that the Lebesgue measure (the unique extension to the Borel sigma algebra of the Lebesgue premeasure on boxes) is translation-invariant.

(b) Prove that $\mathbb{R}^d$ does not admit a translation-invariant Borel *probability* measure μ. In fact, show that for any nonzero vector $\vec{x} \in \mathbb{R}^d$, there does not exist a Borel probability measure that is invariant under translation by $\vec{x}$.

Hint: Find a Borel transversal for the coset equivalence relation of the subgroup $\mathbb{Z}\vec{t} \leqslant \mathbb{R}^d$, i.e. a Borel set $W \subseteq \mathbb{R}^d$ such that $\mathbb{R}^d = \bigsqcup_{n\in\mathbb{Z}}(n\vec{x}+W)$. Do this for $d=1$ first to easily find a set W.

6. In a metric space X, a set $C \subseteq X$ is called a **Cantor set** if it is homeomorphic to the Cantor space $2^{\mathbb{N}}$, i.e. there is a continuous bijection $f : 2^{\mathbb{N}} \to C$ whose inverse is also continuous[1]. In particular, C is a compact subset of X of cardinality continuum. See my short note on Cantor sets [pdf] to learn more about them.

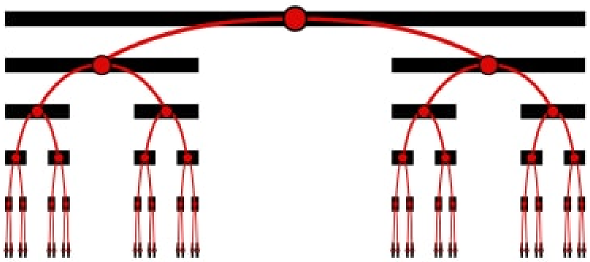
(a) [*Optional*] In a connected[2] metric space X (such as $\mathbb{R}^d$), prove every Cantor set has is compact (hence closed) and has empty interior; in particular, it is nowhere dense.

Hint: If a Cantor set C has nonempty interior U, then there is a further set $V \subseteq U$ such that V is clopen relative to C, i.e. there is an open set $O \subseteq X$ and a closed set $K \subseteq X$ such that $O \cap C = V = K \cap C$. Show that V is clopen in X.

(b) The standard Cantor set in $[0,1]$ is the set $C \coloneqq \bigcap_{n\in\mathbb{N}} \bigcup_{s\in 2^n} C_s$, where each C_s is a closed interval defined inductively by setting $C_\emptyset \coloneqq [0,1]$ and letting C_{s0} and C_{s1} be the bottom third and top third closed subintervals of the closed interval C_s for each $s \in 2^{<\mathbb{N}}$. In particular, $C_0 \coloneqq [0, \frac{1}{3}]$ and $C_1 \coloneqq [\frac{2}{3}, 1]$, $C_{00} \coloneqq [0, \frac{1}{3^2}]$, $C_{01} \coloneqq [\frac{2}{3^2}, \frac{3}{3^2}]$, $C_{10} \coloneqq [\frac{6}{3^2}, \frac{7}{3^2}]$, and $C_{11} \coloneqq [\frac{8}{3^2}, 1]$, etc. Prove that C is indeed a Cantor set (i.e. is homeomorphic to $2^{\mathbb{N}}$) and that C is Lebesgue-null.

[1] The requirement that f^{-1} is continuous is redundant, it is automatically continuous because $2^{\mathbb{N}}$ is compact and X is Hausdorff.

[2] A metric space is **connected** if it has no clopen sets, other than the whole space and $\emptyset$.



Hint: To show that the obvious bijection between $2^{\mathbb{N}}$ and C is a homeomorphism note that the C_s are relatively open in C, hence form a basis for C. For nullness, let U_s be the middle third open interval in C_s, i.e. $U_s := C_s \setminus (C_{s0} \cup C_{s1})$, and calculate $\sum_{s \in 2^{<\mathbb{N}}} \mathrm{lh}(U_s)$.

Remark: Thus, C is an example of a null set of cardinality continuum.

(c) Define a Cantor subset of $[0,1]$ of positive Lebesgue measure.

Hint: Note that in the standard Cantor set, the open interval $U_s := C_s \setminus (C_{s0} \sqcup C_{s1})$ that we remove from C_s occupies $1/3$ of C_s regardless of s. Change the construction so that U_s has length p_n, where $n := \mathrm{lh}(s)$ and the sequence (p_n) goes to 0 fast enough to guarantee $\sum_{n \in \mathbb{N}} 2^n p_n < 1$.

7. Let A be a finite nonempty set and let ν be a probability measure on $\mathcal{P}(A)$ such that $\nu(a) > 0$ for each $a \in A$; e.g. $A := 3 := \{0,1,2\}$ and $\nu(0) := 1/2$, $\nu(1) := 1/3$, and $\nu(2) := 1/6$. Let μ be the Bernoulli measure $\nu^{\mathbb{N}}$ defined in class, i.e. its value on a cylinder $[w]$ is:

$$\mu([w]) := \nu(w_0) \cdot \nu(w_1) \cdots \nu(w_{n-1}),$$

where $w \in A^{<\mathbb{N}}$ and $n := \mathrm{lh}(w)$.

Fix $a_0 \in A$ and prove that $(A \setminus \{a_0\})^{\mathbb{N}}$ is μ-null, i.e. $\mu((A \setminus \{a\})^{\mathbb{N}}) = 0$.

Remark: If $|A| \geqslant 3$, then $(A \setminus \{a_0\})^{\mathbb{N}}$ is another example of a null set of cardinality continuum.

8. [*Optional*] Let $\mathbb{E}_0$ be the equivalence relation on $2^{\mathbb{N}}$ of eventual equality, i.e.

$$x \mathbb{E}_0 y :\Leftrightarrow \forall^\infty n \; x(n) = y(n),$$

where $\forall^\infty n$ means for all large enough n. For each n, let $\sigma_n : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the n^{th} bit flip map, i.e. $\sigma_n(x)$ is the same as x except that its n^{th} coordinate is equal to $1 - x(n)$. Let Γ be the group generated by all the σ_n. Then Γ naturally acts on $2^{\mathbb{N}}$.

(a) Realize that the orbit equivalence relation of this action is exactly $\mathbb{E}_0$.

Remark: Actually, Γ is isomorphic to the direct sum $\oplus_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ so it is a subgroup of the direct product $\prod_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z} \cong 2^{\mathbb{N}}$, hence $\mathbb{E}_0$ is simply the coset equivalence relation of $\oplus_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ as a subgroup of $\prod_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$, just like $E_{\mathbb{Q}}$ is the coset equivalence relation of $\mathbb{Q}$ as a subgroup of $\mathbb{R}$.

(b) Note that the Bernoulli(1/2) measure is invariant under this action, i.e. for any $\mu_{1/2}$-measurable set $A \subseteq 2^{\mathbb{N}}$ and $\gamma \in \Gamma$, we have $\mu_{1/2}(\gamma A) = \mu_{1/2}(A)$.

(c) Prove that every transversal for $\mathbb{E}_0$ is not $\mu_{1/2}$-measurable.

REMARK: In fact, one can also show that every transversal for $\mathbb{E}_0$ is not μ_p-measurable for every $p \in (0,1)$.

9. We say that a real $r \in \mathbb{R}$ admits a **sequence of good rational approximations of exponent** $\alpha > 0$ if there are infinitely many pairs $(p,q) \in \mathbb{Z} \times \mathbb{N}^+$ such that

$$|r - \frac{p}{q}| < \frac{1}{q^\alpha}.$$

Dirichlet's approximation theorem (or rather its immediate consequence) states that every real admits a sequence of good rational approximations of exponent 2.

Prove however that for every $\varepsilon > 0$, almost no real admits a sequence of good rational approximations of exponent $2+\varepsilon$, i.e. the set B of all $r \in \mathbb{R}$ that admit a sequence of good rational approximations of exponent $2+\varepsilon$ is null (with respect to Lebesgue measure).

HINT: First note that it is enough to prove the statement for $[0,1)$ instead of $\mathbb{R}$. Next, express B in terms of the sets

$$A_{p,q} := \left\{ r \in \mathbb{R} : |r - \frac{p}{q}| < \frac{1}{q^{2+\varepsilon}} \right\}$$

where $p, q \in \mathbb{N}^+$ and $p < q$. Finally, what is the measure of $A_{p,q}$?