Last update: 7:07pm (Yerevan), Apr 19.

Mathematical Logic HOMEWORK 8 **Due: Apr 24 (Wed)**

1. Let σ be a signature and $\boldsymbol{A} := (A, \sigma^{\boldsymbol{A}})$ be a σ-structure. Let $P \subseteq A$. Prove that the class of P-definable functions is closed under composition, i.e. if $g : A^k \to A$ and $f_i : A^m \to A$, $i \leqslant k$, are P-definable functions in $\boldsymbol{A}$, then the function $g(f_1, f_2, \ldots, f_k) : A^m \to A$ defined by $\vec{a} \mapsto g(f_1(\vec{a}), f_2(\vec{a}), \ldots, f_k(\vec{a}))$ is P-definable in $\boldsymbol{A}$.

2. For any σ-formula φ, prove $\vdash (\varphi \to \neg\neg\varphi)$. In other words, construct a formal proof of the formula $\varphi \to \neg\neg\varphi$ from the empty theory.

3. Prove the Constant Substitution lemma.

4. As mentioned in class, realize that Compactness theorem is equivalent to the statement that for any σ-theory T and σ-sentence φ, if $T \models \varphi$ then $T_0 \models \varphi$ for some finite subtheory $T_0 \subseteq T$.

5. Prove the following:

(a) (Associativity of +) $\mathrm{PA} \vdash \forall x \forall y \forall z[(x + y) + z = x + (y + z)]$,

(b) $\mathrm{PA} \vdash \forall x(0 + x = x)$,

(c) (Commutativity of +) $\mathrm{PA} \vdash \forall x \forall y(x + y = y + x)$.

REMARK: Good luck!