Last update: 7:21pm (Yerevan), Apr 17.

Mathematical Logic HOMEWORK 7 **Due: Apr 17 (Wed)**

1.* Prove the following theorems, noting first that Lovász's theorem is a special case of Łoś–Tarski. Nonetheless, I suggest proving Lovász's theorem first because it's more approachable.

(a) **Łoś–Tarski Theorem.** Let σ be a signature and $\mathcal{C}$ be a finitely axiomatizable class of σ-structured. Then $\mathcal{C}$ is closed under substructures[1] if and only if $\mathcal{C}$ is axiomatized by a universal σ-sentence.

REMARK: I had originally written the statement of the Łoś–Tarski theorem incorrectly[2], demanding on the left that $\mathcal{C}$ is closed only under finitely generated substructures, but this is false: let $\mathcal{C}$ be the class of all linear orders with a least element[3].

(b) Let $k \in \mathbb{N}$ and call a (potentially infinite) graph k-**coverable** if it admits $\leqslant k$ vertices such that each edge is incident to at least one of them.

Lovász's Theorem. For each $k \in \mathbb{N}$, there exists finitely many finite graphs $H_1, H_2, \ldots, H_m$ (forbidden patterns) such that for every graph G, we have that G is k-coverable if and only if it does not contain any of the H_i as a subgraph.

HINT: First prove that if every finite subcover of a graph G is k-coverable, then such is G. Then take the collection of minimal counter-examples to k-coverability and prove that this collection has to be finite.

REMARK: Lovász's original proof is

2. Let $\sigma_S := (0, S)$ where 0 is a constant symbol and S is a unary function symbol. Let T_S be the σ_S-theory consisting of the following (infinitely-many) axioms:

(S1) Zero has no predecessor: $\forall x (S(x) \neq 0)$.

(S2) The successor function is one-to-one: $\forall x \forall y (S(x) = S(y) \to x = y)$.

(S3) Any nonzero number is a successor of something: $\forall x (x \neq 0 \to \exists y (x = S(y)))$.

(S4$^\infty$) There are no cycles: $\varphi_n := \forall x (S^n(x) \neq x)$ for every $n \in \mathbb{N}$.

(a) Observe that $\boldsymbol{N}_S := (\mathbb{N}, 0, S)$ is a (standard) model of T_S and describe all models of T_S.

(b) Prove that T_S is κ-categorical for each uncountable cardinal κ, and deduce that T_S is complete.

[1] By this we mean that if a σ-structure is in $\mathcal{C}$ then so are all of its substructures.

[2] Thanks Ruben for pointing this out.

[3] Thanks Vahagn for suggesting this example.

3. Let K be a field and let $\overline{K}$ be an algebraic closure of K. A nonconstant polynomial $f \in K[X_1, \dots, X_n]$ is called *irreducible* if whenever $f = gh$ for some $g, h \in K[X_1, \dots, X_n]$, either $\deg(g) = 0$ or $\deg(h) = 0$. Furthermore, f is called *absolutely irreducible* if it is irreducible in $\overline{K}[X_1, \dots, X_n]$ (view f as an element of $\overline{K}[X_1, \dots, X_n]$).

For example, the polynomial $X^2 + 1 \in \mathbb{R}[X]$ is irreducible, but it is not absolutely irreducible since $X^2 + 1 = (X + i)(X - i)$ in $\mathbb{C}[X]$. On the other hand, $XY - 1 \in \mathbb{Q}[X, Y]$ is absolutely irreducible.

Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and prove the following:

Theorem (Noether–Ostrowski Irreducibility Theorem)**.** *For $f \in \mathbb{Z}[X_1, \dots, X_n]$ and prime p, let f_p denote the polynomial in $\mathbb{F}_p[X_1, \dots, X_n]$ obtained by applying the canonical map $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ to the coefficients of f (i.e. mod-ing out the coefficients by p). For all $f \in \mathbb{Z}[X_1, \dots, X_n]$, f is absolutely irreducible (as an element of $\mathbb{Q}[X_1, \dots, X_n]$) if and only if f_p is absolutely irreducible (as an element of $\mathbb{F}_p[X_1, \dots, X_n]$) for all sufficiently large primes p.*

Hint: Your proof should be shorter than the statement of the theorem.

Remark: The original algebraic proof of this theorem is quite involved.