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Mathematical Logic Homework 1 **Due: Feb 21 (Wed)**

Below let σ denote a signature.

1. Design a signature σ_{vs} suitable for vector spaces over $\mathbb{R}$ and define its interpretation in an arbitrary $\mathbb{R}$-vector space $\boldsymbol{V}$.

2. Let $\sigma_{\text{sgp}} := (\cdot)$ be a signature, where $\cdot$ is a binary operation symbol. A σ_{sgp}-structure is called a **semigroup** if $\cdot$ is interpreted as an *associative* binary operation, i.e. for all x, y, z,

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

(a) Give an example of a σ_{sgp}-structure that is **not** a semigroup.

(b) Give an example of a semigroup that is **not** a group (i.e. the interpretation of $\cdot$ is such that there does not exist an inverse or identity).

3. For each of the following examples of σ-structures $\boldsymbol{A}$ and $\boldsymbol{B}$, determine whether $\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$.

(a) $\sigma := (\leqslant)$, where $\leqslant$ is a binary relation symbol, $\boldsymbol{B} := (\mathbb{Z}, \leqslant^{\boldsymbol{B}})$ where $\leqslant^{\boldsymbol{B}}$ is the usual less than or equal to relation on $\mathbb{Z}$, and $\boldsymbol{A} := (2\mathbb{Z}^+, \leqslant^{\boldsymbol{A}})$ where $\mathbb{Z}^+$ is the set of positive integers and $\leqslant^{\boldsymbol{A}}$ is the divisibility relation, i.e. $n \leqslant^{\boldsymbol{A}} m :\Leftrightarrow n \mid m :\Leftrightarrow$ there is $q \in \mathbb{N}^+$ such that $m = qn$.

(b) Same as in (a), except that the underlying set of $\boldsymbol{A}$ is $\{2^n : n \in \mathbb{N}\}$ instead of $2\mathbb{Z}^+$.

(c) $\sigma := \sigma_{\text{rng}} := (0, 1, +, -(), \cdot)$ is the usual signature for rings, $\boldsymbol{B} := (\mathbb{R}, 0, 1, +, -(), \cdot)$ with the usual interpretations, and $\boldsymbol{A} := (\mathbb{Z}, 0, 1, +, -(), \cdot)$ with the usual interpretations.

(d) $\sigma := (f)$ where f is a unary function symbol, $\boldsymbol{B} := (\mathbb{R}, f^{\boldsymbol{B}})$ where $f^{\boldsymbol{B}}(x) := \frac{4}{\pi}\arctan(x)$, $\boldsymbol{A} := (\{0, 1\}, f^{\boldsymbol{A}})$ where $f^{\boldsymbol{A}}(x) := x^2$.

Definition. For a σ-structure $\boldsymbol{B}$ and $A \subseteq B$, we say that A **supports a substructure** of $\boldsymbol{B}$ if there is a substructure $\boldsymbol{A} \subseteq \boldsymbol{B}$ whose universe is A. We also abuse the terminology and simply say that in this case A **is a substructure** even though it is literally incorrect.

4. Let $\boldsymbol{B} := (B, \sigma)$ be a σ-structure and $A \subseteq B$. Prove:

(a) The set A supports at most one substructure of $\boldsymbol{B}$.

(b) The set A supports a substructure of $\boldsymbol{B}$ if and only if A contains all of the constants and is closed under all of the operations of $\boldsymbol{B}$, i.e. $c^{\boldsymbol{B}} \in A$ for each $c \in \text{Const}(\sigma)$, and $f^{\boldsymbol{B}}[A^n] \subseteq A$ for each $n \in \mathbb{N}$ and $f \in \text{Func}(\sigma)$ of arity n.

(c) Conclude that if σ only has relation symbols, then every subset of B supports a substructure.

5. For each of the structures $\boldsymbol{A}$ below, determine the substructure generated by $\emptyset$. You do not have to prove your answer.

(a) $\boldsymbol{A} := (\mathbb{N}, 0, +)$.

(b) $\boldsymbol{A} := (\mathbb{N}, 0, 1, +)$.

(c) $\boldsymbol{A} := (\mathbb{N}, <)$.

(d) $\boldsymbol{A} := (\mathbb{Q}, 0, 1, +, -(), \cdot)$, where $-()$ is the usual additive inverse $x \mapsto -x$.

(e) $\boldsymbol{A} := (\mathbb{Q}, 0, 1, +, -(), \cdot, ()^*)$, where $()^* : \mathbb{Q} \to \mathbb{Q}$ is defined by $x \mapsto x^{-1}$ for $x \neq \emptyset$ and $0 \mapsto 0$.

6. Let $\boldsymbol{A}, \boldsymbol{B}$ be σ-structures and let $h : \boldsymbol{A} \to \boldsymbol{B}$ be a σ-homomorphism. Prove that the image $h(A)$ is a substructure of $\boldsymbol{B}$.

7. Let σ be a signature, $\boldsymbol{A} := (A, \sigma), \boldsymbol{B} := (B, \sigma)$ be σ-structures, and $h : A \to B$ be a function.

(a) Prove that h is a σ-isomorphism if and only if h is a bijective σ-homomorphism satisfying $R^{\boldsymbol{A}}(\vec{a}) \Leftrightarrow R^{\boldsymbol{B}}(h(\vec{a}))$ for each n-ary $R \in \mathrm{Rel}(\sigma)$ and $\vec{a} \in A^n$. Conclude the analogous statement for σ-embeddings.

(b) Deduce that if σ has no relation symbols, then σ-isomorphism is the same as a bijective σ-homomorphism, and σ-embedding is the same as an injective σ-homomorphism.

Remark: That's why this happens with groups, rings, vector spaces, but not with graphs or partial orders.

8. Determine if the following functions are homomorphisms/embeddings/isomorphisms. No justification is necessary. The only acceptable answers are: "not a homomorphism", "homomorphism but not an embedding", "embedding but not an isomorphism", "isomorphism".

If not specified, then the interpretations of the symbols in the signatures are the standard ones.

(i) $n \mapsto 2n$ from $(\mathbb{N}, 0, 1, +)$ to $(\mathbb{N}, 0, 1, +)$.

(ii) $n \mapsto 2n$ from $(\mathbb{N}, 0, +, \cdot)$ to $(\mathbb{N}, 0, +, \cdot)$.

(iii) $n \mapsto 2^n$ from $(\mathbb{N}, 0, +)$ to $\boldsymbol{A} := (\mathbb{N}, 0^{\boldsymbol{A}}, +^{\boldsymbol{A}})$, where $0^{\boldsymbol{A}}$ is 1 and $+^{\boldsymbol{A}}$ is the usual multiplication.

(iv) $(x, y) \mapsto x - y$ from $\boldsymbol{A} := (\mathbb{R}^2, 0^{\boldsymbol{A}}, +^{\boldsymbol{A}})$ to $(\mathbb{R}, 0, +)$, where $0^{\boldsymbol{A}} := (0, 0)$ and $+^{\boldsymbol{A}}$ is the usual coordinate-wise addition of points in $\mathbb{R}^2$, i.e. $(a_0, b_0) + (a_1, b_1) := (a_0 + a_1, b_0 + b_1)$.

(v) $n \mapsto -n$ from $(\mathbb{Z}, 0, +)$ to $(\mathbb{Z}, 0, +)$.

(vi) $n \mapsto n$ from $(\mathbb{N}^+, \leqslant)$ to $\boldsymbol{A} := (\mathbb{N}^+, \leqslant^{\boldsymbol{A}})$, where $\leqslant^{\boldsymbol{A}}$ is the divisibility relation, i.e., $n \leqslant^{\boldsymbol{A}} m$ exactly when n divides m.

(vii) $n \mapsto n$ from $\boldsymbol{A} := (\mathbb{N}^+, \leqslant^{\boldsymbol{A}})$ to $(\mathbb{N}^+, \leqslant)$, where $\leqslant^{\boldsymbol{A}}$ is the divisibility relation, i.e., $n \leqslant^{\boldsymbol{A}} m$ exactly when n divides m.