Last update: 6:35pm, Apr 25.

Metric Spaces and Topology HOMEWORK 9 Due: **Apr 25 (Tue)**

Definition. In a topological space X, we say that a sequence $(x_n) \subseteq X$ **converges** to $x \in X$ if for each open $U \ni x$, we have $\forall^\infty n \; x_n \in U$. In this case, we also say that x is a **limit** of (x_n).

1. Prove that if a topological space is Hausdorff, limits of sequences are unique.

2. Let X be an infinite set and let $\mathcal{T}$ be the cofinite topology, i.e. a set is open if its complement is finite.

(a) Prove that this topology is T_1 but not Hausdorff because any two nonempty open sets intersect.

(b) Let $(x_n) \subseteq X$ be a sequence of pairwise distinct elements. Prove that every point $x \in X$ is a limit of (x_n). In particular, limits are nonunique.

(c) Prove that a subset $D \subseteq X$ is dense if and only if it is infinite.

3. [*Optional*] **Zariski topology.** Let $\mathbb{F}$ be an infinite field (e.g. $\mathbb{F} := \mathbb{Q}, \mathbb{R}, \mathbb{C}$), fix $n \in \mathbb{N}^+$, and consider the Zariski topology on $\mathbb{F}^n$. Prove:

(a) The Zariski topology on $\mathbb{F}^n$ is indeed a topology, namely, finite unions of Zaristki closed sets are Zariski closed.

(b) The intersection of two nonempty Zariski open sets is nonempty. Equivalently, the union of two Zariski closed proper subsets of $\mathbb{F}^n$ is not the whole $\mathbb{F}^n$. Conclude that the Zariski topology is not Hausdorff, although it is T_1.

4. Criterion for a prebasis to be a basis. For a set X, a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis for the topology it generates if and only if $\mathcal{B}$ covers X (i.e. $X = \bigcup \mathcal{B}$) and for each $U, V \in \mathcal{B}$ and $x \in U \cap V$, there is $W \in \mathcal{B}$ with $x \in W \subseteq U \cap V$.

5. For the space $A^{\mathbb{N}}$ with the usual metric d, prove that the sets $[i \mapsto a] := \left\{x \in A^{\mathbb{N}} : x(i) = a\right\}$, $i \in \mathbb{N}$ and $a \in A$, form a prebasis for the topology induced by the metric d.

6. Prove that a topological space is T_1 if and only if every singleton is closed.

7. Let X be a normal (i.e. T_4) topological space and let $U, V \subseteq X$ be open. Prove that if $U \subseteq_c V$, then there is an open set $W \subseteq X$ such that $U \subseteq_c W \subseteq_c V$. (Recall that the notation $A \subseteq_c B$ means $\overline{A} \subseteq B$.)

8. Furstenberg topology on $\mathbb{Z}$. Recall that the Furstenberg topology on $\mathbb{Z}$ is that generated by the cosets $a + b\mathbb{Z}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$.

(a) Prove that the cosets as above, form a basis.

(b) [*Optional*] For $z \in \mathbb{Z}$, let $\|z\| := \sum_{k \nmid z, k \geqslant 1} 2^{-k}$. Prove that this satisfies the triangle inequality $\|x+y\| \leqslant \|x\| + \|y\|$ for all $x, y \in \mathbb{Z}$ (and hence $\|\cdot\|$ is a norm). Deduce that $d(x,y) := \|x-y\|$ is a metric on $\mathbb{Z}$.

(c) [*Optional*] Prove that this metric d induces the Furstenberg topology on $\mathbb{Z}$.

Remark: Thus, $\mathbb{Z}$ with this topology is metrizable, zero-dimensoinal, perfect, and countable. Such a space is *unique* up to homeomorphism (by Sierpiński's theorem). In particular, $\mathbb{Z}$ with the Furstenberg topology is homeomorphic to $\mathbb{Q}$ (with the relative topology of $\mathbb{R}$).